NONLINEAR LOVE WAVES IN ELASTIC MEDIA

BY

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To My Family

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Blessed be the name of the Lord God almighty who has blessed us with all spiritual blessings in heavenly places. I am so grateful for His will, mercies and protection throughout the preparation of this work. This thesis would not have been completed successfully without the help and support from some important people. I would like to acknowledge their help and efforts in a unique and special way. Firstly, I would like to express my profound and heartfelt gratitude to my thesis adviser, Prof. Fiazud Din Zaman for his selfless, amazing support and significant comments throughout the preparation of this work. I always say I was fortunate to have an adviser of his caliber whose love and passion for the success of his students are his highest priorities. His immense support and fatherly advise has brought me this far. It is my earnest prayer that the good Lord grant him more years on earth to continue his good works. I also pray that one day I would become an excellent adviser to my students as Prof. Zaman has been to me. I am also indebted to my co-adviser, Prof. Mustapha Kassem for his encouragement and help. He was always there for me anytime I needed his help throughout my studies and also during the preparation of this thesis. I say God richly bless you Prof. Kassem. I would also acknowledge the help and guidance *from my thesis committee members: Dr. Faisal Fairag, Dr. Ahmed Bonfoh and Dr. Muhammed Yousuf. I am grateful for their suggestions and relevant inputs during my research. My deepest gratitude to my parents, siblings and friends for their unfailing love and assistance. Their prayers kept me on my toes to work harder. Finally, I would like to acknowledge KFUPM for supporting this thesis.*

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THESIS ABSTRACT

NAME:	Samuel Opoku Agyemang
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The horizontally polarized shear elastic waves are of interest in seismology and engineering. A. E. H Love proved the existence of such waves in a homogeneous elastic layer overlying an elastic half-space. In this thesis, we first discuss Love waves in an isotropic layer overlying an inhomogeneous elastic half-space and obtain the dispersion relation in the layer. The dependence of phase velocity on the wave number is displayed graphically. Next, we consider an anisotropic layer overlying an anisotropic half-space. The dispersion relation satisfied by Love wave is obtained in this case. This dispersion relation reduces to the one for isotropic case. Also for an inhomogeneous anisotropic layer overlying an anisotropic half-space we obtain the dispersion relation in a determinant form. The case of Love waves in the nonlinear model of the media is studied in the end. The stress-strain relation now involves a potential. We use the Murnaghan model and we consider a *typical Love wave problem with added assumption of nonlinear deformation. The resulting nonlinear wave equation in terms of the displacements with both linear parts and nonlinear parts is studied using perturbation method.*

الملخص

الإسم : صمويل أبوكيو اجيمانج

عنوان الدراسة: موجات لوف غير الخطية في الأوساط المرنة

مجال التخصص : الرياضيات

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تحظى موجات القص المرنة ذات الاستقطاب الأفقي بأهمية في علم الزلازل والهندسة. وقد أثبت أ. لوف وجود هذه الموجات في الطبقات المرنة المتجانسة والمتداخلة مع نصف الفضاء المرن. في بداية هذه الأطروحة ستتم مناقشة موجات لوف عندما تحدث في نصف المجال المتجانس المرن بالإضافة إلى التوصل لمعادلة التشتت في هذه الطبقة ذات الخصائص غير الاتجاهية. هذا وسيتم عرض اعتماد السرعة الطورية على الرقم الموجي من خلال الرسومات البيانية. بعد ذلك، سيتم اعتبار الطبقات ذات الخصائص الاتجاهية المتداخلة مع نصف الفضاء الاتجاهي وسيتم التوصل إلى معادلة التشتت الموافقة لموجة لوف لهذا النوع من الموجات. معادلة التشتت للطبقات ذات الخصائص المتجهة سيتم اختزالها إلى معادلة التشتت للطبقات ذات الخصائص غير المتجهة. إضافة إلى ذلك، سيتم بصورة محددة اشتقاق معادلة التشتت للطبقات غير المتجانسة ذات الخصائص الاتجاهية والمتداخلة مع الفضاء الاتجاهي. في نهاية هذه الأطروحة سيتم دراسة موجات لوف باستخدام نموذج رياضي غير خطي للوسط تحت الدراسة. علاقة التوتر والإجهاد في هذا الدراسة أصبحت تتضمن طاقة كامنة. هذا وقد تم استخدام نموذج مورنجان وتم اعتبار مشكلة موجة لوف بصيغتها الاعتيادية مع إضافة افتراض أن التشوه الحاصل للطبقة غير خطي. المعادلة الموجية غير الخطية الناتجة من هذا النموذج سيتم التعبير عنها باستخدام الإزاحة بجزئيها الخطي والغير خطي وسيتم حلها باستخدام طريقة الإضافة.

CHAPTER 1

INTRODUCTION

An elastic solid is a material which undergoes small deformation when external force or stress is applied and comes back to its original configuration when such a force is removed. In linear elasticity, the deformation is assumed to be small and linear rate of deformation is used to define the strain. The generalized Hooke's law which states that the stress is directly proportional to the strain provides the constitutive equations. In elastic waves, the material particles exhibit to and fro motion in three planes with respect to the direction of wave propagation. This gives rise to three types of elastic waves: horizontally polarized waves (SH-waves), longitudinally polarized waves (P-waves) and vertically polarized waves (SV-waves). We focus on the horizontally polarized SH waves that travel along the free surface of the elastic half-space. In a half-space model without any over-lying layer, we find that the Rayleigh surface waves are propagated which consist of P and SV components [1],[55]. However, no horizontally polarized wave is propagated in this half-space model. The SH component of earthquakes observed in seismographs was resolved by Love [44], who proposed that a surface wave of SH type can travel in an elastic layer with different elastic moduli overlying a half-space.

Deforming elastic materials are classified into hypoelastic materials, Cauchy elastic materials and hyperelastic materials. Hyperelastic constitutive laws are used to model materials that undergo a very large strain when subjected to an external stress [59]. Hypoelastic constitutive laws are used to model materials that even under small strains exhibits nonlinear, but reversible, stress strain behavior. Cauchy elastic constitutive laws are used to model materials that exhibits linear, but reversible, stress strain relation when subjected to small strains. In elastic materials, the strain is small and the Hooke's law holds.

All these types of elastic materials are characterized by strain rate, stress and strains produced due to applied forces. In linear elasticity, Hooke's law holds and the second order stress tensor, called the Lagrange-Cauchy stress tensor is used to describe the stress within the body. The corresponding strain is represented by the Cauchy-Green strain tensor. However, in case when Hooke's law is no longer applicable, the Piola-Kirchhoff stress tensor along with the Cauchy-Green strain tensor or Almansi strain tensor are used.

In chapter one of this thesis, we study elastic materials, anisotropic materials, isotropic materials and surface waves. We also study linear Love waves in isotropic materials. In chapter two, Love waves in homogeneous isotropic materials , solution in the layer and half-space are studied. Dispersion relation in the layer is obtained by the Greens function approach. Numerical results were obtained from the dispersion relation in the layer and graphs of phase velocity against wave-number were plotted. We discussed the effects of inhomogeneities associated with Love wave problem. In chapters three and four, linear Love waves in the most general anisotropic materials are studied and the dispersion equations are derived in determinant form. In chapter five, nonlinear Love wave is analyzed and the perturbation method is employed to solve the nonlinear Love waves derived from the Murnaghan model up to the second approximation.

1.1 Anisotropic Elastic Materials

Hooke's Law states that the force required to extend or compress a spring by some distance is directly proportional to that distance. We consider the generalized Hooke's law in Cartesian coordinates which states that, the stress in an elastic body is proportional to the strain. Mathematically it can be stated as

$$\sigma_{qr} = C_{qrlm}\varepsilon_{lm} \qquad q, r, l, m = 1, 2, 3. \tag{1.1}$$

σ_{qr} denotes the second order stress tensor, C_{qrlm} denotes a 4^{th}- order elastic stiffness tensor of material properties and ε_{lm} denotes the second order strain tensor. In linear elasticity we assume that, the strains of the elastic material are infinitesimally small which gives a relationship between strain tensor $\varepsilon = (\varepsilon_{lm})$ and components of the displacement vector $\mathbf{u} = (u_m)$ which represents the classical

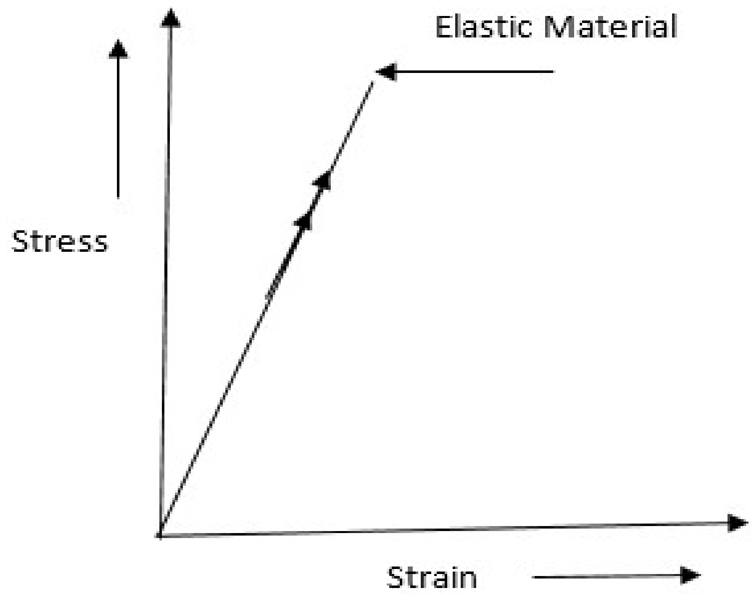

Figure 1.1: Stress-strain curve for an elastic material.

Cauchy relation :

$$\varepsilon_{lm} = \frac{1}{2}(u_{m,l} + u_{l,m}) \tag{1.2}$$

The 4^{th}-order stiffness tensor of materials has 81 material constants for three-dimensional problems and 16 material constants for two-dimensional problems. Because of the symmetry nature of the stress and strain tensors, we realize that the elastic stiffness tensor must satisfy the relations

$$C_{qrlm} = C_{rqlm} = C_{qrml} = C_{rqml}$$

and consequently the 81 constants reduce to 36 independent constants. Symmetry in strain energy density further lessens the number of material elastic moduli to 21. We proceed to write the constitutive equation for a linear elastic material taking into account these symmetries as follows:

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}
$$

Based on the presence of symmetries in the structure of the materials, different types of models of anisotropy are considered in literature. The structure of the stiffness tensor is determined by these symmetries [46]. Each type of symmetry results in the invariance of the stiffness tensor to a special symmetry transformation, (rotation about specific axes and reflection about specific planes) [46]. For applications in seismology, engineering and building construction the most commonly used models are orthotropic materials, transversely isotropic materials and isotropic materials [57]. Other material symmetries which are anisotropic such as triclinic, monoclinic materials, tetragonal materials, cubic materials and many others are studied intensively in crystallography [57]. In what follows we give a brief description of orthotropic, tranversely isotropic and isotropic elastic materials.

1.1.1 Orthotropic Elastic Materials

Orthotropic materials are materials in which elastic properties are symmetric with respect to three perpendicular axes. That is, there are three mutually orthogonal planes of symmetry. A typical example of such material is wood. They are basically known to have nine independent elastic stiffness constants [32], [33], [34].

1.1.2 Transversely Isotropic Elastic Materials

Transversely isotropic materials have one plane in which material properties remain the same. The material properties are symmetric about an axis that is normal to the plane of isotropy (xy-plane). They have three mutually orthogonal planes of reflection symmetry and axial symmetry with respect to the z-axis. A typical example of this type of material is plywood and layered materials. They are defined with five independent coefficients [33], [34], [57] [62].

1.2 Isotropic Elastic Materials

The elastic properties in isotropic materials are the same in all directions. Typical examples are glass, metals and alloys. Isotropic materials are characterized by two independent elastic constants. These constants are independent of the choice of coordinate system [57].

The most general isotropic 4^{th}-order elastic stiffness tensor is denoted by;

$$C_{qrlm} = \lambda\delta_{qr}\delta_{lm} + \mu(\delta_{ql}\delta_{rm} + \delta_{qm}\delta_{rl}) \tag{1.3}$$

where

$$C_{1111} = C_{2222} = C_{3333} = \lambda + 2\mu \quad ; C_{1212} = C_{1313} = C_{2323} = \lambda \quad ; C_{4444} =$$

$$C_{5555} = C_{6666} = \frac{1}{2}(C_{1111-C_{1212}}) = \mu$$

Here μ and λ are termed as the Lame constants. This gives us an elastic stiffness tensor of the form:

$$C_{iklm} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & \textbf{symmetry} & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}$$

We note that, from isotropic elastic stiffness tensor, the generalized Hooke's law for the isotropic material becomes

$$\sigma_{qr} = \lambda \varepsilon_{ll} \delta_{qr} + 2\mu \varepsilon_{qr} \tag{1.4}$$

1.3 Surface Waves

Surface waves are mechanical waves in which the disturbance in the medium is significant near the surface. The amplitude of motion of the particle decreases sharply away from the surface or interface of the elastic medium [57].

Surface waves are sometimes called seismic waves when they propagate freely along the earth surface. This type of waves have lower frequencies than body waves when they both travel through the earth crust. The two most studied elastic surface waves are Rayleigh and Love waves.

1.3.1 Love waves

Love surface waves proposed by Love in 1911 [43] have many applications in seismology, geophysics and earthquake engineering. They are horizontally polarized SH dispersive waves that propagate in the elastic layer overlying an elastic half-space with dissimilar elastic properties [57]. Love waves only travel in a low velocity surface layer as compared to the half-space. Love waves travel faster than the other type of surface waves, known as Rayleigh waves. Thus in seismographs the Love waves appear to arrive first. The figure below represents Love waves.

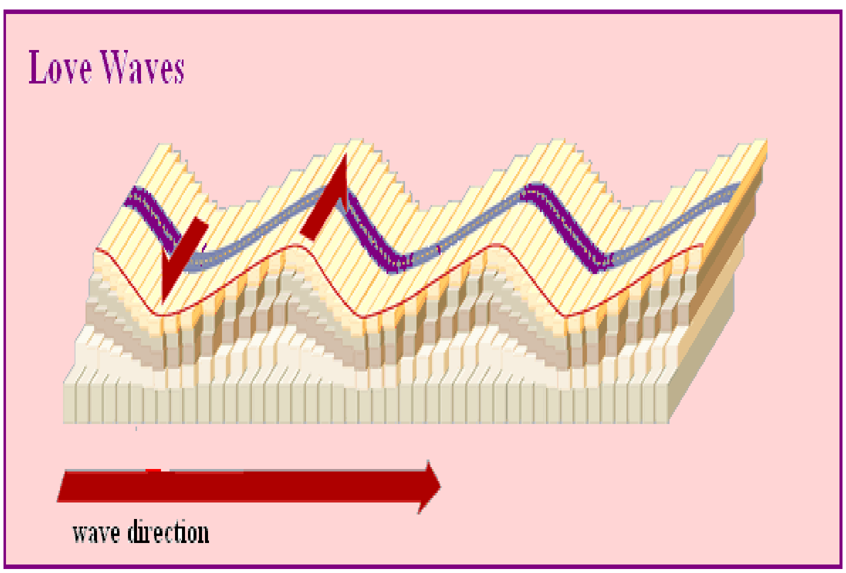

Figure 1.2: Diagram of Love waves.

1.4 Linear Love wave in isotropic materials

1.4.1 Problem formulation

We consider an elastic, homogeneous and isotropic layer of uniform thickness H, overlying an elastic, homogeneous and isotropic half-space. The coordinate axes $Oxyz$ are chosen such that $z - axis$ is directed downwards. The half-space occupies $z > 0$ while the layer occupies $-H \leq z \leq 0$. The density and the elastic parameters for the layer are given by ρ_l , λ_l, and μ_l respectively while the parameters for the half-space are represented by ρ_h, λ_h, and μ_h respectively.

Now we consider the horizontally polarized shear wave for which both longitudinal and vertical displacements u_1 and u_2 respectively are zero. Hence the only possibility of propagation of the wave is along the $Oz - axis$ direction near the interface between half-space and layer. The wave is thus represented as

$$u_3^{l(h)} = U_3^{l(h)}(z)e^{i(kx-\omega t)} \tag{1.5}$$

We know from equation (1.5) that only two components, shear stresses σ_{31} and σ_{32} are non-zero. Here we notice that there is a simple motion consisting of only one component (u_3) leaving us with one wave equation in the layer and one in the half-space as;

$$\left[\mu_{l(h)}\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right) - \rho_{l(h)}\frac{\partial^2}{\partial t^2}\right]u_3^{l(h)}(x,z,t) = 0 \tag{1.6}$$

That is

$$\frac{\partial^2 u_3^{l(h)}}{\partial z^2} + \frac{\partial^2 u_3^{l(h)}}{\partial x^2} = \frac{1}{(v_T^{l(h)})^2}\frac{\partial^2 u_3^{l(h)}}{\partial t^2}.$$

Substituting (1.5) into (1.6) results in transforming partial differential equations (1.6) into ordinary differential equations:

$$\left(U_{(1)3}^{l(h)}\right)''_{,11} + k^2\Big[\Big(\frac{v}{v_T^{l(h)}}\Big)^2 - 1\Big]U_{(1)3}^{l(h)} = 0; v_T^{l(h)} = \sqrt{\frac{\mu_{l(h)}}{\rho_{l(h)}}} \tag{1.7}$$

We therefore obtain

$$\Big[\frac{d^2}{dz^2} + k^2\Big[\Big(\frac{v}{v_T^{l(h)}}\Big)^2 - 1\Big]\Big]U_3^{l(h)}(z) = 0 \tag{1.8}$$

We now look for the solution in the layer and half-space where the wave is localized (restricted) around the interface $z = 0$.

1.4.2 Solution in the half-space

Considering the equation

$$\Big[\frac{d^2}{dz^2} + k^2\Big[\Big(\frac{v}{v_T^{h}}\Big)^2 - 1\Big]\Big]U_3^{h(1)}(z) = 0 \tag{1.9}$$

Solving the ordinary differential equations (1.9) we have,

$$\frac{d^2 U_3^{h(1)}}{dz^2} + k^2(\beta_h)^2 U_3^{h(1)}(z) = 0$$

$$U_3^{h(1)} = l_h e^{-\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}kz} + B_h e^{\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}kz}.$$

Where $(\beta_h)^2 = 1 - \left(v/v_T^h\right)^2$ and $k_{Love} = \omega/v_{Love}$. The solution in the half-space is found to be

$$U_3^{h(1)} = L_h e^{-\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}kz} \tag{1.10}$$

in which we impose a condition that $(\beta_h)^2 = 1 - \left(v/v_T^h\right)^2 > 0$. That is, we impose a condition that both the root and radicand should be positive. The velocity (v_T^h) of the plane horizontally polarized transverse wave in the half-space is greater than the velocity of the Love wave. Here l_h is a constant and unknown amplitude factor.

1.4.3 Solution in the layer

The solution in the layer is given as

$$U_3^{l(1)}(z) = A_{1l} sin\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}kz + A_{2l} cos\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}kz. \tag{1.11}$$

Where $\beta_l = \sqrt{\left(v/v_T^l\right)^2 - 1}$, A_{1l} and A_{2l} are constants and unknown amplitude factors. The root and radicand $(\beta_l)^2 = (v/v_T^l)^2 - 1 > 0$ must be positive as in that of half-space. Thus the velocity (v_T^l) of plane horizontally polarized tranverse wave in the layer is less than the velocity of linear Love wave or the condition means that the phase velocity of the shear wave in the layer must be less than the phase velocity of Love wave. We note that $k = k_{Love}$ and $v = v_{Love}$ are the wave number and the velocity of linear Love wave respectively [57], [62].

Solution in both the layer and half-space as in (1.10) and (1.11) is obtained subject to the condition of existence of Love waves as

$$v_T^h > v_{Love} > v_T^l \tag{1.12}$$

Hence the solution in the above problem of linear Love wave propagation is written interms of three unknown amplitude constants as below

$$u_3^{h(1)}(x,z,t) = l_h e^{-\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}kz} e^{i(kx-\omega t)}, \quad x \in (-\infty,\infty), z \in [0,\infty) \tag{1.13}$$

$$u_3^{l(1)} = \left\{ A_{1l} sin\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}kz + A_{2l} cos\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}kz \right\} e^{i(kx-\omega t)}, \tag{1.14}$$
$$x \in (-\infty,\infty), z \in [-H,0).$$

1.4.4 Boundary conditions

To obtain full solution to the linear Love wave problem we solve for the three unknown amplitude constants l_h, A_{1l} and A_{2l} by considering three boundary conditions. That means we consider the boundary conditions on the planes $z = -H$ and $z = 0$. On the Plane $z = -H$, the normal component of the stress vanishes so that $\left(\partial U_3^{(l)}/\partial z\right)_{z=-H} = 0$. This gives

$$\left(\partial U_3^{(l)}/\partial z\right)_{z=-H} = A_{1l} cos\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}kH + A_{2l} sin\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}kH$$

$$= A_{1l} cos(k\beta_l H) + A_{2l} sin(k\beta_l H) = 0 \tag{1.15}$$

where $\beta_l = \sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}$. On the plane $z = 0$ the condition of full mechanical contact at the interface implies continuity of displacement and the normal component of stress. This gives, $U_3^l(0) = U_3^h(0)$; $\mu_l\left(\partial U_3^{(l)}/\partial z\right)_{z=-0} = \mu_h\left(\partial U_3^{(h)}/\partial z\right)_{z=+0}$. Using the first condition we get

$$U_3^{l(1)}(0) = A_{1L} sin\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}k(0) + A_{2l} cos\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}k(0)$$

$$U_3^{l(1)}(z) = A_{2l}.$$

Also

$$U_3^{h(1)}(z) = l_h e^{-\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}kz}$$

$$U_3^{h(1)}(0) = l_h e^{-\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}k(0)}$$

$$U_3^{h(1)}(0) = l_h.$$

This implies that

$$A_{2l} = l_h.$$

Now for the last boundary condition, $\mu_l\left(\partial U_3^{(l)}/\partial z\right)_{z=-0} = \mu_h\left(\partial U_3^{(h)}/\partial z\right)_{z=+0}$

We have that,

$$\mu_l\left(\partial U_3^{(l)}/\partial z\right)_{z=0} = \mu_l k\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}A_{1l}$$

Also,

$$\mu_h\left(\partial U_3^{(h)}/\partial z\right)_{z=0} = -\mu_h l_h k\sqrt{\left[1 - \left(v/v_T^h\right)^2\right]}.$$

This gives that

$$\mu_l k \sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]} A_{1l} = -\mu_h l_h k \sqrt{\left[1 - \left(v/v_T^h\right)^2\right]}$$

That is

$$\mu_l \beta_l A_{1l} = -\mu_h \beta_h A_{2l}$$

Now from

$$A_{1l} cos(k\beta_l H) + A_{2l} sin(k\beta_l H) = 0,$$

We have that

$$A_{1l} cos(k\beta_l H) = -A_{2l} sin(k\beta_l H)$$

So

$$\frac{A_{1l}}{A_{2l}} = -tan(k\beta_l H).$$

Simplifying further we have,

$$\tfrac{A_{1l}}{A_{2l}} = -tan\left[\sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]} kH\right]$$

We have already seen that,

$$\frac{A_{1l}}{A_{2l}} = -\frac{\mu_h \beta_h}{\mu_l \beta_l}$$

And

$$\frac{A_{1l}}{A_{2l}} = -\frac{\mu_h \sqrt{\left[1 - \left(v/v_T^h\right)^2\right]}}{\mu_l \sqrt{\left[\left(v/v_T^l\right)^2 - 1\right]}}$$

Thus

$$\frac{\mu_h\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}}{\mu_l\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}}=tan\left\{\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}kH\right\} \tag{1.16}$$

The equation (1.16) is the dispersion relation for Love wave. We see that one of the amplitudes is arbitrary which signifies that the wave under consideration is a running surface wave and to determine the wave number and phase velocity it is important to solve the transcedental equation

$$\frac{\mu_h\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}}{\mu_l\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}}=tan\left\{\frac{\omega H}{v}\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}\right\}. \tag{1.17}$$

Where $k = \omega/v_{Love}$. The transcedental equation (1.17) shows that there is a nonlinear dependence of the phase velocity of Love waves on frequency.

This is a feature of dispersion of Love waves. We note that the transcedental equation (1.17) has countable number of roots. The infinite number of wave modes and wave numbers are generated by the infinite number of roots. Since the amplitude constants are inter-dependent.

$$A_{1l}=-\frac{\mu_h\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}}{\mu_l\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}}A_{2l}$$

Form $A_{2l} = l_h$ we have;

$$A_{1l}=-\frac{\mu_h\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}}{\mu_l\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}}l_h$$

Hence the solution in the layer and the half space is given as

$$u_3^{h(1)}(x,z,t) = l_h e^{-\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}kz} e^{i(kx-\omega t)}, \quad x \in (-\infty,\infty), z \in [0,\infty) \tag{1.18}$$

$$u_3^{l(1)}(x,z,t) = l_h \Bigg\{ -\frac{\mu_h\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}}{\mu_l\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}} sin\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}kz$$
$$+cos\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}kz\Bigg\} e^{i(kx-\omega t)},\ x \in (-\infty,\infty), z \in [-H,0). \tag{1.19}$$

1.5 Literature Review

Love surface waves in elastic media is a classical topic. Many authors have extensively studied it, especially in relation to anisotropic and isotropic media. A lot has been done on the solutions to both classical linear elastic Love waves and non-linear elastic Love waves. Dey, et al (2004) [21] in their work presented a study on Love waves propagating in a poro-elastic layer overlying a poro-elastic half-space. The study revealed that such a medium transmits two types of Love waves. Farnell and Adler [23] talked about waves propagating in thin layers. They discussed in details both Love waves and Rayleigh waves in anisotropic and isotropic media.

Ke, et al. (2005) [41], in their work dealt with Biot's theory for tranversely isotropic fluid saturated porous media. They also derived dispersion Love wave equations in a transversely isotropic media with an inhomogenous layer under consideration. They solved this equation by an iterative method. Negi and Upadhyay (1967) [48] studied the effect of inhomogeneity and anisotropy on Love wave dispersion. Here the inhomegeneous layer was bounded on either side by homogeneous, isotropic solid half spaces. Saha, et. al (2015) [70] in their recent work studied Love waves in a heterogeneous orthotropic layer under changeable initial stress over a gravitating porous half-space. They derived the dispersion equation of Love waves in a closed form using the variable separation approach. Yanson (2002) [76] in his work studied Love waves that are concentrated in the neighbourhood of the surface of anisotropic elastic body. The author dealt with the solvability of the construction of the uniform asymptotics for Love waves as in Azhotkin and Babich (1990) [8].

In 1911, Love [43] investigated a dispersive surface wave (Love wave). Love [43], [44] in his work investigated that Love waves are shear waves that are polarized horizontally in a homogeneous isotropic linearly elastic half-space, enclosed by a layer of uniform thickness that has various mechanical properties with a displacement perpendicular to the plane of propagation. Many authors including Lamb, Stonely, Bleustein, Gulyaev, further studied other kinds of elastic waves which were later named after them. Love [44] in his early work talked about dispersion of Love and Rayleigh waves and talked extensively on elastic solid half-space which is being covered by a single solid layer.

McCall in 1994 [47] made a theoretical study of nonlinear elastic wave propagation. In this work the author studied wave equation for an isotropic, homogeneous, elastic solid with cubic anharmonicity in the moduli, accounting for the attenuation by introducing complex linear and nonlinear moduli. He used the Green's function approach to solve governing equations. Ohnabe and Nowinski (1979) [52] in their work studied Love waves in an elastic isotropic half-space with superficial layer of a different incompressible material. Rushchitskii (1996) [69] applied the slowly varying amplitudes approach to the equations describing plane waves in both one and two phases and analyzed the second harmonic generating problem. In 1885, Rayleigh [55] in his early work studied extensively presence of surface waves propagating on a free plane boundary of an isotropic homogeneous linear elastic half-space. He also studied their relevance in seismography. Other authors including Kalyanasundaram et al. [35] [36],[37] ,[38] have done extensive investigations on nonlinear dispersion of both Rayleigh and Love waves.

In 2003, Cattani and Rushchitskii [14] made a study on cubically nonlinear elastic waves. They described models mainly on hyperelastic materials such as the Murnaghan model and various modifications on the Murnaghan potential. In their work they also considered three basic methods for solving wave equations which are the method of successive approximation, the slowly varying amplitude approach and the wavelet-based method. Rushchitsky and Khotenko (2012)[66] made a study on Rayleigh wave propagating along the boundary of surface of an elastic half-space whose nonlinear deformation is described by the Murnaghan model. In their work, they derived six new nonlinear wave equations and three simple equations of nonlinear motion for displacement. They solved the resulting nonlinear wave equations by the iterative method up to the second approximation. In 2013, Rushchitsky [59] studied the classical nonlinear Love waves equation from a five-constant Murnaghan model and he applied the Iterative method to solve the resulting nonlinear elastic Love wave equation. In this work the author derived a new nonlinear equation that helps in finding the wave number. He only considered up to the second approximation and neglected third and higher approximations. Rushchitsky (2015) [58] in his recent work dealt extensively with both linear and nonlinear equations elucidating surface wave propagating on a free flat boundary of an elastic half-space that is in an anti-plane strain state. He considered nonlinearity in both physical and geometrical, by a Murnaghan model. The author considered four surface waves : Harmonic and simple waves in nonlinear and linear case and there was an inconsistency amid fundamental assumptions and final result in each case, and he concluded that surface waves was impossible to be described in these cases. In 1999, Romeo [56] used an integral method to show the well posedness of the generalized Love's problem in elastic anisotropic media. The solution was derived using the iterative approach where the dispersion equation was obtained by imposing the pertinent boundary conditions.

Garcia-Reimbert and Minzoni in 1988 [2] made a study on some nonlinear effects on Love waves. The authors studied the evolution of a narrow band package of weakly Love waves which propagate in an elastic layer of thickness H superimposed on an elastic half space, where the elastic properties of the layer are assumed to be different from those of the half space.

In 2013, Gupta et. al [30] studied the propagation of Love waves in non-homogeneous substratum over initially stressed heterogeneous half space and derived the dispersion equation of phase velocity. Kakar R. and Kakar S. (2012) [40] investigated the propagation of Love waves in a non-homogeneous elastic media. They also used the method of variable separation and the substituting method for solving second order partial differential equation to find general solution to the problem. They then discussed the dispersion of Love waves. Ghorai et. al (2010) [28] studied the propagation of Love waves in a fluid-saturated porous layer under a rigid boundary and lying over an elastic half-space under gravity. they developed the equation of motion for different media employing appropriate boundary conditions at the interface of porous layer, elastic half space under gravity and rigid layer. They also studied the effects of porosity and gravity of the layers in the propagation of Love waves .

Pan and Chakrabarty (1972) [53] studied Love waves in inhomogeneous anisotropic elastic solids.They investigated the existence of Love waves in non-homogeneous and tranvsersely isotropic elastic layer overlying a semi-infinite isotropic elastic solid. The authors derived the frequency equation and calculated the velocity of such waves for different layer thickness numerically. Kalyanasundaram (1988) [39] used a multiscale perturbution method to study the nonlinear mode coupling between Rayleigh and Love wave on a half space of homogeneous isotropic elastic solid with a thin superficial layer of another such solid. Ahmed and Abo-Dahab (2010) in their work studied the propagation of Love waves in an orthotropic granular layer under initial stress overlying a semi-infinite granular medium. They employed the Fourier transform method in finding the dispersion equation [5]. Abd-Alla and Ahmed (1999) [4] studied Love waves propagating in a non-homogeneous orthotropic elastic medium under changeable initial stress. They employed the Fourier transform method for finding the dispersion equation. Recently, Vaishnav et. al (2016) [72] studied the propagation of Love-type wave in an initially stressed porous medium over a semi-infinite orthotropic medium with rectangular irregular interface. They employed the method of separation of variables in finding the dispersion relation of Love-type wave. They also went further to obtain the dispersion equations in the classical form.

Chattopadhyay et. al [16] in their paper studied Love waves in a porous Layer overlying an inhomogeneous half-space due to a point source. The Green's function technique was applied in deriving the dispersion relation for Love waves in the porous layer. Chattopadhyay et. al [17] considered propagation of Love waves in an homogeneous medim lying over and inhomogeneous half-space. They again employed the Green's function developed by Ghosh to obtain solve the problem and obtained the dispersion relation for the layer. In 2012 Chattopadhyay et. all [18] studied the propagation of SH waves in a homogeneous viscoelastic isotropic layer lying over a semi-infinite heterogeneous viscoelastic isotropic half-space due to point source. In their work they assumed the inhomogeneity parameters associated to rigidity, internal friction and density to be functions of depth. They employed the Green's function technique to obtain the dispersion relation of the SH wave. Chattopadhyay et. all [19] investigated the propagation of SH wave due to a point source in a magnetoelastic self-reinforced layer overlying a heterogeneous self-reinforced half space. They used the Green's Function to derive the dispersion in the closed form. In 1990 Asghar et. al [6] studied Love -type waves in a vertically inhomogeneous intermediate layer. Zaman et. al in 1990 [77] used Fourier transform and the Green's function method to derive the dispersion relation of Love-Type wave in an inhomogeneous layer trapped between two half-spaces due to a line source. Zaman et. al (1991) [78] used the Green's function approach to study the dispersion of Love waves in an non-homogeneous layer due to a point source. In this thesis, we will study the perturbation method involved in nonlinear elastic Love wave equation.

CHAPTER 2

LOVE WAVES IN A LAYERED MODEL DUE TO A POINT SOURCE

In this chapter, we describe the propagation of Love waves in a homogeneous elastic layer overlying an inhomogeneous elastic half-space. Ghosh (1970) [29] and Zaman (1991) [78] have used the perturbation and the Green's function approach to deal with inhomogeneities in the half-space. We present here the problem in which the half-space has a density varying linearly with depth [78] and derive a dispersion relation in the layer. This will provide necessary background to our work dealing with anisotropic model.

2.1 Formulation of the problem

We consider the problem in which a homogeneous isotropic layer of finite thickness H overlies an inhomogeneous isotropic elastic half space. We choose the origin of coordinates at a point on the free surface, $(z = 0)$, the $x-$axis along the free surface and the $z-$axis is taken vertically downward. A harmonic point source is assumed to be situated at a point $S(0, H)$ on the interface of the layer and the half-space. Subscripts 1 and 2 refer to the layer $(0 \leq z \leq H)$ and the half space $(z \geq H)$ respectively.

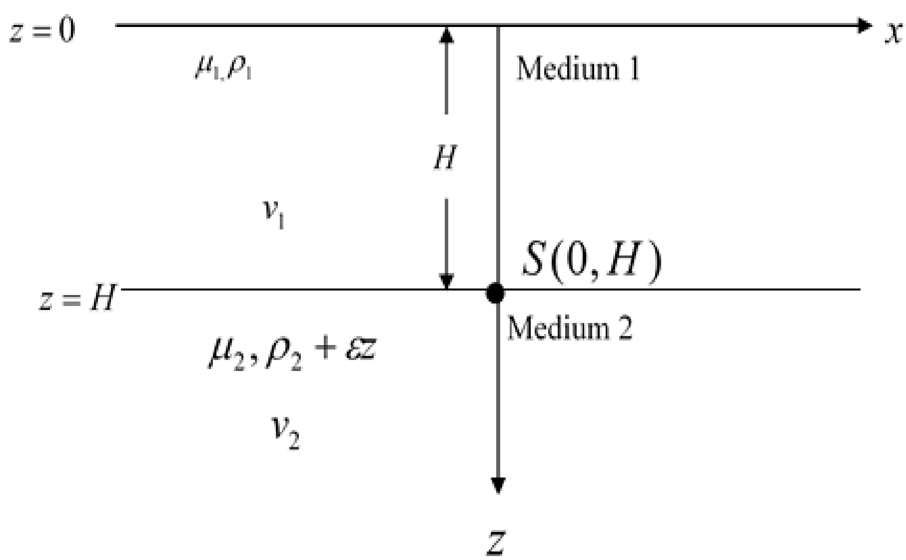

Figure 2.1: Geometry of the problem I.

Let μ_1 and μ_2 be the moduli of rigidity for the layer (medium 1) and the half space (medium 2) respectively. Let ρ_1 be the density of the layer and $\rho_2 + \varepsilon z$ is the density of the half space, where εz measures the inhomogeneity. The equations of motion are given below as

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = \rho_1 \frac{\partial^2 v_1}{\partial t^2} + 4\pi\sigma_1(r,t) \tag{2.1}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = (\rho_2 + \varepsilon z)\frac{\partial^2 v_2}{\partial t^2} \tag{2.2}$$

Where the inhomogeneous term in (2.1) appears due to the presence of the source density $\sigma_1(r,t)$. We shall take $\sigma_1(r,t) = \delta(x)\delta(z-H)e^{i\omega t}$ which represents a time harmonic source with angular frequency ω. From the generalized Hooke's law for isotropic materials in equation (1.4), the most general isotropic $4^{th}-$order elastic stiffness tensor in equation (1.3) and the linear strain tensor in equation (1.2), the equation of motion in medium 1 (2.1) can be written as

$$\frac{\partial}{\partial x}\Big(\mu_1 \frac{\partial v_1}{\partial x}\Big) + \frac{\partial}{\partial z}\Big(\mu_1 \frac{\partial v_1}{\partial z}\Big) = \rho_1 \frac{\partial^2 v_1}{\partial t^2} + 4\pi\sigma_1(r,t)$$

$$\mu_1 \frac{\partial^2 v_1}{\partial x^2} + \mu_1 \frac{\partial^2 v_1}{\partial z^2} = \rho_1 \frac{\partial^2 v_1}{\partial t^2} + 4\pi\delta(x)\delta(z-H)$$

Dividing throughout by μ_1 we have,

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} = \frac{\rho_1}{\mu_1}\frac{\partial^2 v_1}{\partial t^2} + \frac{4\pi}{\mu_1}\delta(x)\delta(z-H).$$

Replacing v_1 by $v_1 e^{i\omega t}$ and $4\pi\delta(x)\delta(z-H)$ by $4\pi\delta(x)\delta(z-H)e^{i\omega t}$ we have

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + \frac{\rho_1}{\mu_1}\omega^2 v_1 = \frac{4\pi}{\mu_1}\delta(x)\delta(z-H),$$

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} + k_1^2 v_1 = \frac{4\pi}{\mu_1}\delta(x)\delta(z-H). \tag{2.3}$$

Similarly for medium 2, Equation (2.2) can be written as

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + k_2^2 v_2 = -\frac{\varepsilon}{\mu_2}\omega^2 z v_2. \tag{2.4}$$

Where $k_j^2 = \frac{\rho_j}{\mu_j}\omega^2$ for $j = 1$ and 2.

2.2 Boundary Conditions

The boundary conditions at the interface and the free surface are

$$\frac{\partial v_1}{\partial z} = 0 \qquad at \quad z = 0, \tag{2.5}$$

$$v_1(z) = v_2(z) \qquad at \quad z = H, \tag{2.6}$$

$$\mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z} \qquad at \quad z = H. \tag{2.7}$$

2.3 Solution to the problem

We shall assume the time dependence to be e^{iwt} and can be suppressed throughout. We find the solution of (2.3) and (2.4) using the Fourier transform $V(\xi, z)$ and its inverse $v(x, z)$ given below as

$$\begin{aligned} V(\xi, z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} v(x, z) e^{i\xi x} dx \\ v(x, z) &= \int_{-\infty}^{\infty} V(\xi, z) e^{-i\xi x} d\xi \end{aligned} \tag{2.8}$$

using these transforms, the equations of motion in (2.3) and (2.4) reduces to

$$\frac{d^2 V_1}{dz^2} - \alpha^2 V_1 = \frac{2}{\mu_1}\delta(z - H) = 4\pi\sigma_1(z) \tag{2.9}$$

$$\frac{d^2V_2}{dz^2} - \beta^2 V_2 = -\frac{\varepsilon}{\mu_2}\omega^2 z V_2 = 4\pi\sigma_2(z) \tag{2.10}$$

Where $\alpha^2 = \xi^2 - k_1^2$ and $\beta^2 = \xi^2 - k_2^2$. Fourier transform of the boundary conditions given in equations (2.5), (2.6) and (2.7) are,

$$\frac{dV_1}{dz} = 0 \qquad at \quad z = 0, \tag{2.11}$$

$$V_1(z) = V_2(z) \qquad at \quad z = H, \tag{2.12}$$

$$\mu_1 \frac{dV_1}{\partial z} = \mu_2 \frac{dV_2}{dz} \qquad at \quad z = H. \tag{2.13}$$

Now let $G_1(z, z_0)$ be Green's function for the layer $(0 \leq z \leq H)$ satisfying $\frac{dG_1}{dz} = 0$ at $z = 0$ and at $z = H$. $G_1(z, z_0)$ is the solution of inhomogeneous equation,

$$\frac{d^2G_1(z, z_0)}{dz^2} - \alpha^2 G_1(z, z_0) = \delta(z - z_0), \tag{2.14}$$

Where z_0 is a point in the layer $(0 \leq z \leq H)$. Multiplying (2.9) by $G_1(z, z_0)$ and (2.14) by $V_1(z)$, we have

$$G_1(z, z_0)\frac{d^2V_1}{dz^2} - \alpha^2 V_1 G_1(z, z_0) = 4\pi\sigma_1(z)G_1(z, z_0), \tag{2.15}$$

$$V_1 \frac{d^2G_1(z, z_0)}{dz^2} - \alpha^2 V_1 G_1(z, z_0) = \delta(z - z_0)V_1. \tag{2.16}$$

Subtracting the two equations (2.15) and (2.16) and integrating from 0 to H we obtain

$$\int_0^H \left[G_1(z, z_0)\tfrac{d^2V_1}{dz^2} - V_1 \tfrac{d^2G_1(z,z_0)}{dz^2}\right]dz = \int_0^H 4\pi\sigma_1(z)G_1(z, z_0)dz - \int_0^H \delta(z - z_0)V_1 dz$$

$$\int_0^H \frac{d}{dz}\Big[G_1(z,z_0)\frac{dV_1}{dz} - V_1\frac{dG_1(z,z_0)}{dz}\Big]dz = \int_0^H 4\pi\sigma_1(z)G_1(z,z_0)dz - V_1(z_0).$$

Using the boundary conditions $\frac{dG_1}{dz} = 0$ at $z = 0$ and $z = H$ and from $\left[\frac{dV_1}{dz}\right]_{z=0} = 0$ we have

$$\Big[G_1(z,z_0)\frac{dV_1}{dz} - V_1\frac{dG_1(z,z_0)}{dz}\Big]_0^H = \int_0^H 4\pi\sigma_1(z)G_1(z,z_0)dz - V_1(z_0)$$

$$G_1(H,z_0)\Big[\frac{dV_1}{dz}\Big]_{z=H} = \int_0^H 4\pi\sigma_1(z)G_1(z,z_0)dz - V_1(z_0),$$

Where we take $4\pi\sigma_1(z) = \frac{2}{\mu_1}\delta(z-H)$ we obtain,

$$G_1(H,z_0)\Big[\frac{dV_1}{dz}\Big]_{z=H} = \frac{2}{\mu_1}G_1(H,z_0) - V_1(z_0)$$

Interchanging z_0 and z and remembering the symmetry nature of Green's function, $G_1(H,z) = G_1(z,H)$ we have,

$$V_1(z) = \frac{2}{\mu_1}G_1(z,H) - G_1(z,H)\Big[\frac{dV_1}{dz}\Big]_{z=H} \tag{2.17}$$

Again similarly, let $G_2(z,z_0)$ be the Green's function for the half space $z \geq H$, satisfying the boundary conditions $\frac{dG_2}{dz} = 0$ at $z = H$ and as $z \to \infty$

$$\frac{d^2G_2(z,z_0)}{dz^2} - \beta^2 G_2(z,z_0) = \delta(z-z_0) \tag{2.18}$$

Where z_0 is a point in the half space $(H \leq z \leq \infty)$. Multiplying (2.10) by $G_2(z,z_0)$ and (2.18) by V_2 we get

$$G_2(z,z_0)\frac{d^2V_2}{dz^2} - \beta^2 V_2 G_2(z,z_0) = 4\pi\sigma_2(z)G_2(z,z_0) \tag{2.19}$$

$$V_2\frac{d^2G_2(z,z_0)}{dz^2} - \beta^2 V_2 G_2(z,z_0) = \delta(z-z_0)V_2 \tag{2.20}$$

Subtracting the two equations (2.19) and (2.20) and integrating from H to ∞ we obtain,

$$\int_H^\infty \left[G_2(z,z_0)\tfrac{d^2V_2}{dz^2} - V_2\tfrac{d^2G_2(z,z_0)}{dz^2}\right]dz = \int_H^\infty 4\pi\sigma_1(z)G_2(z,z_0)dz - \int_H^\infty \delta(z-z_0)V_2 dz,$$

$$\int_H^\infty \frac{d}{dz}\Big[G_2(z,z_0)\frac{dV_2}{dz} - V_2\frac{dG_2(z,z_0)}{dz}\Big]dz = \int_H^\infty 4\pi\sigma_2(z)G_2(z,z_0)dz - V_2(z_0).$$

Using the boundary conditions $\dfrac{dG_1}{dz} = 0$ at $z = H$ and $z \to \infty$ we have

$$\Big[G_2(z,z_0)\frac{dV_2}{dz} - V_2\frac{dG_2(z,z_0)}{dz}\Big]_H^\infty = \int_H^\infty 4\pi\sigma_2(z)G_2(z,z_0)dz - V_2(z_0)$$

$$-G_2(H,z_0)[\frac{dV_2}{dz}]_{z=H} = \int_H^\infty 4\pi\sigma_2(z)G_2(z,z_0)dz - V_2(z_0)$$

Interchanging z_0 and z, due to symmetry of Green's function $G_2(H,z) = G_2(z,H)$ and $G_2(z,z_0) = G_2(z_0,z)$, we obtain

$$V_2(z) = G_2(z,H)\left[\frac{dV_2}{dz}\right]_{z=H} + \int_H^\infty 4\pi\sigma_2(z_0)G_2(z,z_0)dz_0 \tag{2.21}$$

Applying the transformed boundary conditions in equations (2.12) and (2.13) that is the boundary conditions at the interface $z = H$

$$V_1(H) = V_2(H)$$

and

$$\mu_1\left[\frac{dV_1}{dz}\right]_{z=H} = \mu_2\left[\frac{dV_2}{dz}\right]_{z=H}$$

Using the these conditions we have

$$V_1(H) = \frac{2}{\mu_1}G_1(H,H) - G_1(H,H)\Big[\frac{dV_1}{dz}\Big]_{z=H}.$$

$$V_2(H) = G_2(H,H)\Big[\frac{dV_2}{dz}\Big]_{z=H} + \int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0$$

We have that

$$\tfrac{2}{\mu_1}G_1(H,H) - G_1(H,H)\Big[\tfrac{dV_1}{dz}\Big]_{z=H} = G_2(H,H)\Big[\tfrac{dV_2}{dz}\Big]_{z=H} + \textstyle\int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0$$

$$\Big(G_1(H,H) + (\mu_1/\mu_2)G_2(H,H)\Big)\Big[\tfrac{dV_2}{dz}\Big]_{z=H} = \tfrac{2}{\mu_1}G_1(H,H) - \textstyle\int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0$$

Where $\Big[\frac{dV_2}{dz}\Big]_{z=H} = \frac{\mu_1}{\mu_2}\Big[\frac{dV_1}{dz}\Big]_{z=H}$

$$\Big[\frac{dV_1}{dz}\Big]_{z=H} = \frac{1}{B}\left\{\frac{2}{\mu_1}G_1(H,H) - \int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0\right\}$$

Where $B = G_1(H,H) + (\mu_1/\mu_2)G_2(H,H)$

So

$$\Big[\frac{dV_1}{dz}\Big]_{z=H} = \left\{\frac{2G_1(H,H)}{\mu_1(G_1(H,H) + (\mu_1/\mu_2)G_2(H,H))} - \frac{\int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0}{G_1(H,H) + (\mu_1/\mu_2)G_2(H,H)}\right\} \tag{2.22}$$

Put (2.22) into (2.17) we obtain,

$$V_1(z) = \frac{2}{\mu_1}G_1(z,H) - G_1(z,H)\left\{\frac{2G_1(H,H)}{\mu_1(G_1(H,H) + (\mu_1/\mu_2)G_2(H,H))} - \frac{\int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0}{G_1(H,H) + (\mu_1/\mu_2)G_2(H,H)}\right\}$$

Simplifying the first two terms on the right hand side(RHS) and replacing $4\pi\sigma_2(z_0)$ by $-\frac{\varepsilon}{\mu_2}\omega^2 z_0 V_2(z_0)$, we have

$$V_1(z) = \frac{2G_1(z,H)G_2(H,H)}{\mu_2(G_1(H,H) + \mu_1 G_2(H,H))} - \varepsilon\frac{\omega^2 G_1(z,H)\int_H^\infty z_0 V_2(z_0)G_2(H,z_0)dz_0}{\mu_2 G_1(H,H) + \mu_1 G_2(H,H)} \tag{2.23}$$

Equation (2.23) is an integral equation and $V_1(z)$ may be determined from this equation by successive substitutions. As a first approximation we neglect terms involving ε to obtain

$$V_1(z) = \frac{2G_1(z,H)G_2(H,H)}{\mu_2(G_1(H,H) + \mu_1 G_2(H,H))} \tag{2.24}$$

We note that $V_1(z)$ is completely determined by (2.23) provided $G_1(z, z_0)$ and $G_2(z, z_0)$ are known.

We find $V_2(z)$ in a similar manner as follows,

$$V_1(H) = \frac{2}{\mu_1}G_1(H,H) - G_1(H,H)\Big[\frac{dV_1}{dz}\Big]_{z=H},$$

$$V_2(H) = G_2(H,H)\Big[\frac{dV_2}{dz}\Big]_{z=H} + \int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0$$

We have that,

$$\tfrac{2}{\mu_1}G_1(H,H) - G_1(H,H)\Big[\tfrac{dV_1}{dz}\Big]_{z=H} = G_2(H,H)\Big[\tfrac{dV_2}{dz}\Big]_{z=H} + \textstyle\int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0$$

$$\Big(G_2(H,H) + (\mu_2/\mu_1)G_1(H,H)\Big)\Big[\tfrac{dV_2}{dz}\Big]_{z=H} = \tfrac{2}{\mu_1}G_1(H,H) - \textstyle\int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0$$

Where $\Big[\frac{dV_1}{dz}\Big]_{z=H} = \frac{\mu_2}{\mu_1}\Big[\frac{dV_2}{dz}\Big]_{z=H}$.

$$\Big[\frac{dV_2}{dz}\Big]_{z=H} = \frac{1}{C}\left\{\frac{2}{\mu_1}G_1(H,H) - \int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0\right\}$$

Where $C = G_2(H,H) + (\mu_1/\mu_2)G_1(H,H)$.

So

$$\Big[\frac{dV_2}{dz}\Big]_{z=H} = \left\{\frac{2G_1(H,H)}{\mu_1(G_2(H,H) + (\mu_1/\mu_2)G_1(H,H))} - \frac{\int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0}{G_2(H,H) + (\mu_2/\mu_1)G_1(H,H)}\right\} \tag{2.25}$$

Putting (2.25) into (2.21) we have;

$$V_2(z) = \frac{2G_1(H,H)G_2(z,H)}{\mu_1(G_2(H,H) + (\mu_1/\mu_2)G_1(H,H))} - \Big\{\frac{G_2(z,H)}{G_2(H,H) + (\mu_2/\mu_1)G_1(H,H)}\Big\}\times$$
$$\int_H^\infty 4\pi\sigma_2(z_0)G_2(H,z_0)dz_0 + \int_H^\infty 4\pi\sigma_2(z_0)G_2(z,z_0)dz_0$$

Replacing $4\pi\sigma_2(z_0) = -\frac{\varepsilon}{\mu_2}\omega^2 z_0 V_2(z_0)$ we obtain

$$V_2(z) = \frac{2G_1(H,H)G_2(z,H)}{\mu_1(G_2(H,H) + \mu_2 G_1(H,H)} + \varepsilon\Big\{\frac{\omega^2 G_2(z,H)}{\mu_2(G_2(H,H) + (\mu_2/\mu_1)G_1(H,H))}\Big\}\times$$
$$\int_H^\infty z_0 V_2(z_0)G_2(H,z_0)dz_0 - \varepsilon\frac{\omega^2}{\mu_2}\int_H^\infty z_0 V_2(z_0)(G_2(z,z_0)dz_0 \tag{2.26}$$

Here $V_2(z)$ is to be determined from (2.26) by the method of successive approximations. The value of $V_2(z)$ obtained from (2.26) when inserted in (2.23) gives the value of $V_1(z)$. Because we are keen in determining the value of $V_1(z)$, which will give the displacement at any point in the layer and when the first approximation is needed we can neglect ε and take

$$V_2(z) = \frac{2G_1(H,H)G_2(z,H)}{\mu_1(G_2(H,H) + \mu_2 G_1(H,H)} \tag{2.27}$$

which will certainly give the displacement at any point in the half space when taken as homogeneous. Substituting $V_2(z)$ from (2.27) into $V_1(z)$ we get

$$V_1(z) = \frac{2G_1(z,H)G_2(H,H)}{\mu_2(G_1(H,H)+\mu_1 G_2(H,H))} - \left\{\varepsilon\frac{\omega^2 G_1(z,H)}{\mu_2 G_1(H,H)+\mu_1 G_2(H,H)}\right\}\times \int_H^\infty \frac{2G_1(H,H)G_2(z,H)z_0}{\mu_1(G_2(H,H)+\mu_2 G_1(H,H)}G_2(H,z_0)dz_0 \tag{2.28}$$

The value of $V_1(z)$ can be evaluated from (2.28) provided we know the values of $G_1(z,H)$, $G_2(z,H)$, $G_2(z,z_0)$ and $G_1(z,z_0)$ is a solution of the differential equation (2.14) [17]. In order to determine these Green's functions we follow the procedure outlined by [71][77][78]. We need to solve equation (2.14). Two linearly independent solutions of the equation

$$\frac{d^2Q}{dz^2} - \alpha^2 Q = 0 \qquad z \neq z_0 \tag{2.29}$$

that vanish at $z = \infty$ and $z = -\infty$ are $Q_1(z) = e^{\alpha z}$ and $Q_2(z) = e^{-\alpha z}$ respectively. The solutions of (2.29) for an infinite medium is therefore

$$\frac{Q_1(z)Q_2(z_0)}{U} \qquad z < z_0$$

and

$$\frac{Q_1(z_0)Q_2(z)}{U} \qquad z > z_0$$

Where

$$U = Q_1(z)Q_2'(z) - Q_2(z)Q_1'(z) = -2\alpha$$

So

$$\frac{Q_1(z)Q_2(z_0)}{U} = -\frac{1}{2\alpha}e^{(z-z_0)} \qquad z < z_0$$

and

$$\frac{Q_1(z_0)Q_2(z)}{U} = -\frac{1}{2\alpha}e^{-(z-z_0)} \qquad z > z_0$$

Therefore the solution of equation (2.14) for an infinite medium is $-\frac{1}{2\alpha}e^{-\alpha|z-z_0|}$. Since $G_1(z, z_0)$ is to satisfy the conditions $\frac{dG_1(z, z_0)}{dz} = 0$ at both $z = H$ and $z = 0$. We assume

$$G_1(z, z_0) = A_1 e^{\alpha z} + A_2 e^{-\alpha z} - \frac{1}{2\alpha}e^{-\alpha|z-z_0|}. \tag{2.30}$$

Using the above conditions we have that,

$$G_1(z, z_0) = -\frac{1}{2\alpha}\Bigg[e^{-\alpha|z-z_0|} + e^{\alpha z}\left\{\frac{e^{-\alpha(H+z_0)} + e^{-\alpha(H-z_0)}}{e^{\alpha H} - e^{-\alpha H}}\right\} + e^{-\alpha z}\left\{\frac{e^{\alpha(H-z_0)} + e^{-\alpha(H-z_0)}}{e^{\alpha H} - e^{-\alpha H}}\right\}\Bigg] \tag{2.31}$$

So

$$G_1(z, H) = -\frac{1}{\alpha}\left\{\frac{e^{\alpha z} + e^{-\alpha z}}{e^{\alpha H} - e^{-\alpha H}}\right\} \tag{2.32}$$

$$G_1(H, H) = -\frac{1}{\alpha}\left\{\frac{e^{\alpha H} + e^{-\alpha H}}{e^{\alpha H} - e^{-\alpha H}}\right\} \tag{2.33}$$

Similarly, $G_2(z, z_0)$ can be shown as

$$G_2(z, z_o) = -\frac{1}{2\gamma}\left[e^{-\gamma|z-z_0|} + e^{-\gamma(z+z_0-2H)}\right] \tag{2.34}$$

So that

$$G_2(H, z_0) = -\frac{1}{\gamma} e^{-\gamma(z_0 - H)} \tag{2.35}$$

$$G_2(H, H) = -\frac{1}{\gamma} \tag{2.36}$$

Now inserting the values of $G_1(z, H)$, $G_1(H, H)$, $G_2(H, z_0)$ and $G_2(H, H)$ from the above relations into (2.28) we have

$$\begin{aligned} V_1(z) = &-\frac{2(e^{\alpha z} + e^{-\alpha z})}{\mu_2\gamma(e^{\alpha H} + e^{-\alpha z}) + \mu_1\alpha(e^{\alpha H} - e^{-\alpha H})} + \\ &\varepsilon \frac{2\omega^2(e^{\alpha z} + e^{-\alpha z})(e^{\alpha H} + e^{-\alpha H})}{\Big[\mu_2\gamma(e^{\alpha H} + e^{-\alpha z}) + \mu_1\alpha(e^{\alpha H} - e^{-\alpha H})\Big]^2} \Big[\frac{H}{2\gamma} + \frac{1}{4\gamma^2}\Big], \end{aligned} \tag{2.37}$$

$$\begin{aligned} V_1(z) = &-\frac{2(e^{\alpha z} + e^{-\alpha z})}{\Big[e^{\alpha H}(\mu_2\gamma + \mu_1\alpha) - e^{-\alpha H}(\mu_1\alpha - \mu_2\gamma)\Big]} \Bigg[1 - \\ &\varepsilon \frac{\omega^2(e^{\alpha H} + e^{-\alpha H})}{\Big[e^{\alpha H}(\mu_2\gamma + \mu_1\alpha) - e^{-\alpha H}(\mu_1\alpha - \mu_2\gamma)\Big]} \Big[\frac{H}{2\gamma} + \frac{1}{4\gamma^2}\Big]\Bigg]. \end{aligned}$$

Where $D_1 = e^{\alpha H}(\mu_1\alpha + \mu_2\gamma)$ and $D_2 = e^{-\alpha H}(\mu_1\alpha - \mu_2\gamma)$. This gives

$$V_1(z) = -\frac{2(e^{\alpha z} + e^{-\alpha z})}{D_1 - D_2} \Big[1 - \varepsilon \frac{\omega^2(e^{\alpha H} + e^{-\alpha H})}{D_1 - D_2} \Big[\frac{H}{2\gamma} + \frac{1}{4\gamma^2}\Big]\Big].$$

$V_1(z)$ may be approximated to

$$V_1(z) = \frac{-2(e^{\alpha z} + e^{-\alpha z})}{D_1 - D_2 + \varepsilon\omega^2(e^{\alpha H} + e^{-\alpha H})\Big(\frac{H}{2\gamma} + \frac{1}{4\gamma^2}\Big)}.$$

Simplifying further gives,

$$V_1(z) = \frac{-2(e^{\alpha z} + e^{-\alpha z})}{D_1 - D_2 + \varepsilon\omega^2\Big(\frac{H}{\gamma} + \frac{1}{2\gamma^2}\Big) cosh(\alpha H)}$$

The corresponding displacement $v_1(x,z)$ at a point in the layer is obtained from taking the Fourier inverse of $V_1(z)$ as

$$v_1(x,z) = -2\int_{-\infty}^{\infty} \frac{(e^{\alpha z}+e^{-\alpha z})e^{-i\xi x}d\xi}{(D_1-D_2)+\varepsilon\omega^2\left(\frac{H}{\gamma}+\frac{1}{2\gamma^2}\right)cosh(\alpha H)} \tag{2.38}$$

Where the time factor is neglected. The dispersion relation of Love waves is thus obtained by equating the denominator of the integral (2.38) to zero. That is;

$$(D_1-D_2)+\varepsilon\omega^2\Big(\frac{H}{\gamma}+\frac{1}{2\gamma^2}\Big)cosh(\alpha H) = 0$$

$$e^{\alpha H}(\mu_1\alpha+\mu_2\gamma) - e^{-\alpha H}(\mu_1\alpha-\mu_2\gamma)) + \varepsilon\omega^2\Big(\frac{H}{\gamma}+\frac{1}{2\gamma^2}\Big)cosh(\alpha H) = 0$$

Now replacing α by $i\alpha_2$ we have

$$e^{i\alpha_2 H}(i\mu_1\alpha_2+\mu_2\gamma) - e^{-i\alpha_2 H}(i\mu_1\alpha_2-\mu_2\gamma)) + \varepsilon\omega^2\Big(\frac{H}{\gamma}+\frac{1}{2\gamma^2}\Big)cosh(i\alpha_2 H) = 0$$

$$itan(i\alpha_2 H) + \frac{\mu_2\gamma}{\mu_1\alpha_2} + \varepsilon\frac{\omega^2}{2\mu_1\alpha_2}\Big(\frac{H}{\gamma}+\frac{1}{2\gamma^2}\Big)\frac{cosh(i\alpha_2 H)}{cos(i\alpha_2 H)} = 0$$

Since $itan(i\alpha_2) = -tan(\alpha_2 H)$ and $cosh(i\alpha_2 H) = cos(i\alpha_2 H)$ we have

$$tan(\alpha_2 H) = \frac{\mu_2\gamma}{\mu_1\alpha_2} + \varepsilon\omega^2\left(\frac{H}{\gamma}+\frac{1}{2\gamma^2}\right) \tag{2.39}$$

Now since $\alpha = i\alpha_2$, $c_1^2 = \frac{\mu_1}{\rho_1}$ replacing ξ by k and $kc = \omega$ we have $\alpha_2 = k\left(\frac{c^2}{c_1^2}-1\right)^{1/2}$. Also $\gamma = k\left(1-\frac{c^2}{c_2^2}\right)^{1/2}$ since $c_2^2 = \frac{\mu_2}{\rho_2}$. We thus have from (2.39) that;

$$tan\left[kH\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}\right] = \frac{\mu_2\sqrt{\left(1-\frac{c^2}{c_2^2}\right)}}{\mu_1\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}} + \frac{\varepsilon H}{4\rho_1}\left\{\frac{2(\frac{c^2}{c_1^2})}{\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}\sqrt{\left(1-\frac{c^2}{c_2^2}\right)}} + \frac{\frac{c^2}{c_1^2}}{kH\sqrt{\left(\frac{c^2}{c_1^2}-1\right)\left(1-\frac{c^2}{c_2^2}\right)}}\right\}, \tag{2.40}$$

Which is the dispersion relation of Love waves of the problem under consideration. Further putting $\varepsilon \to 0$ we get the classical dispersion for Love wave [1].

$$tan\left[kH\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}\right] = \frac{\mu_2\sqrt{\left(1-\frac{c^2}{c_2^2}\right)}}{\mu_1\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}} \tag{2.41}$$

For the real Love wave in medium 1 we require that $c_1 < c < c_2$.

2.4 Numerical Results, Graphical representation and Discussions

For numerical calculations, we consider the problem under consideration in Section 2.1 and the corresponding dispersion equation is given as what we have in (2.40). For graphical purposes we have taken data from [17] where $\dfrac{\mu_2}{\mu_1} = 1.8$ and $\dfrac{c_1}{c_2} = \dfrac{3.7}{4.5}$. When $\dfrac{\varepsilon H}{4\rho_1}$ takes on the values 0.0, 0.2 and 0.5, the values of kH for different values of $\dfrac{c}{c_1}$ are given in the table and graphs plotted below. In the figure curves are plotted to exhibit the variation in wave number, inhomogeneity and Love wave velocity. Figure 3.2 depicts phase velocity (c/c_1) against wave number (kH) for different values of $s = \frac{\varepsilon H}{4\rho_1}$. The curves show that the phase velocity decreases for increasing wave number.

		kH (Wave number)	
	$s = \frac{\varepsilon H}{4\rho_1} = 0$	$s = \frac{\varepsilon H}{4\rho_1} = 0.2$	$s = \frac{\varepsilon H}{4\rho_1} = 0.5$
$\frac{c}{c_1}$(Phase Velocity)			
1.03	5.34	5.76	5.99
1.05	3.85	4.29	4.53
1.07	3.03	3.51	3.75
1.09	2.48	2.99	3.25
1.11	2.06	2.62	2.89
1.13	1.71	2.33	2.61
1.15	1.37	2.09	2.33
1.17	1.14	1.91	2.20
1.19	0.82	1.73	2.07
1.21	0.38	1.55	1.94

Table 2.1: This table shows the values of phase velocity and wave number at different values of inhomogeneous parameter.

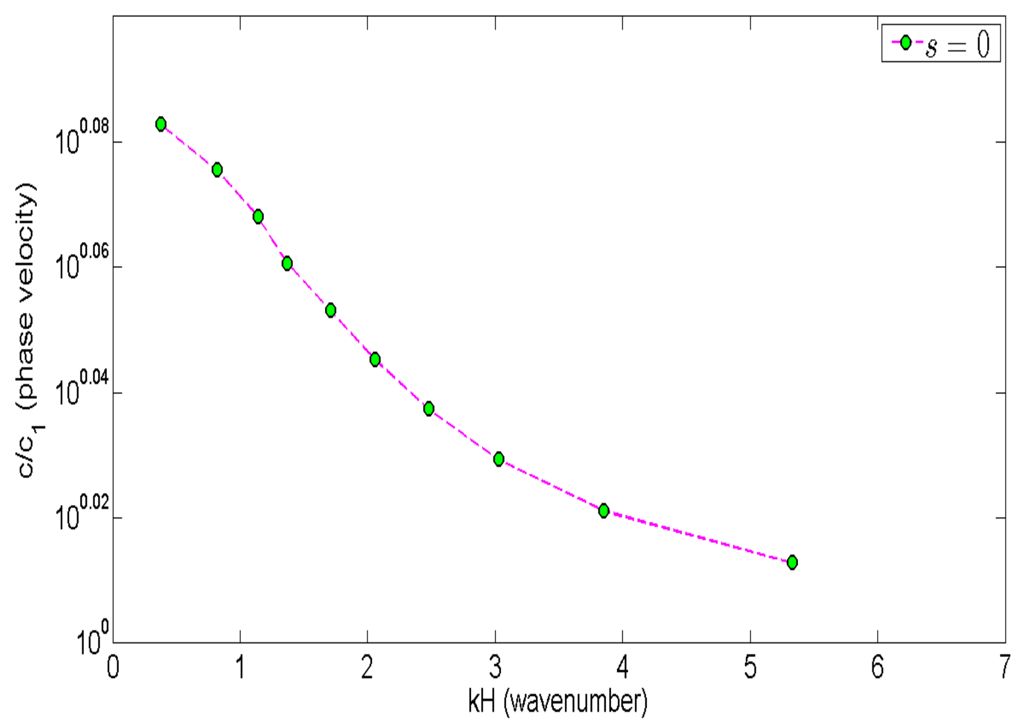

Figure 2.2: Phase velocity curve for Love waves in homogeneous case.

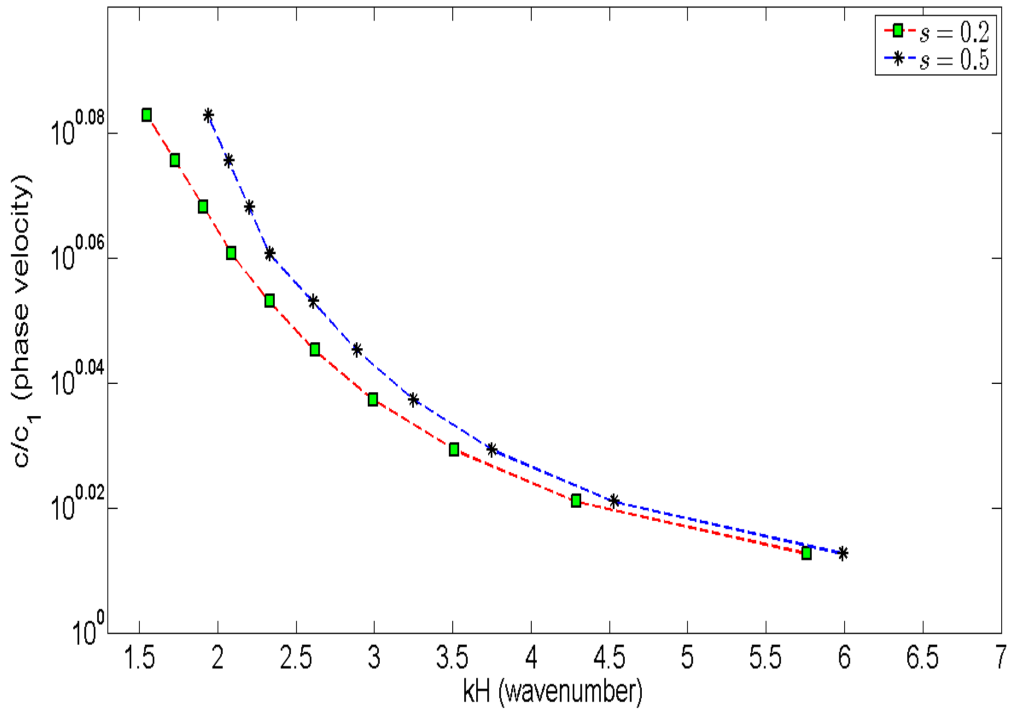

Figure 2.3: Phase velocity curve for Love waves in inhomogeneous cases.

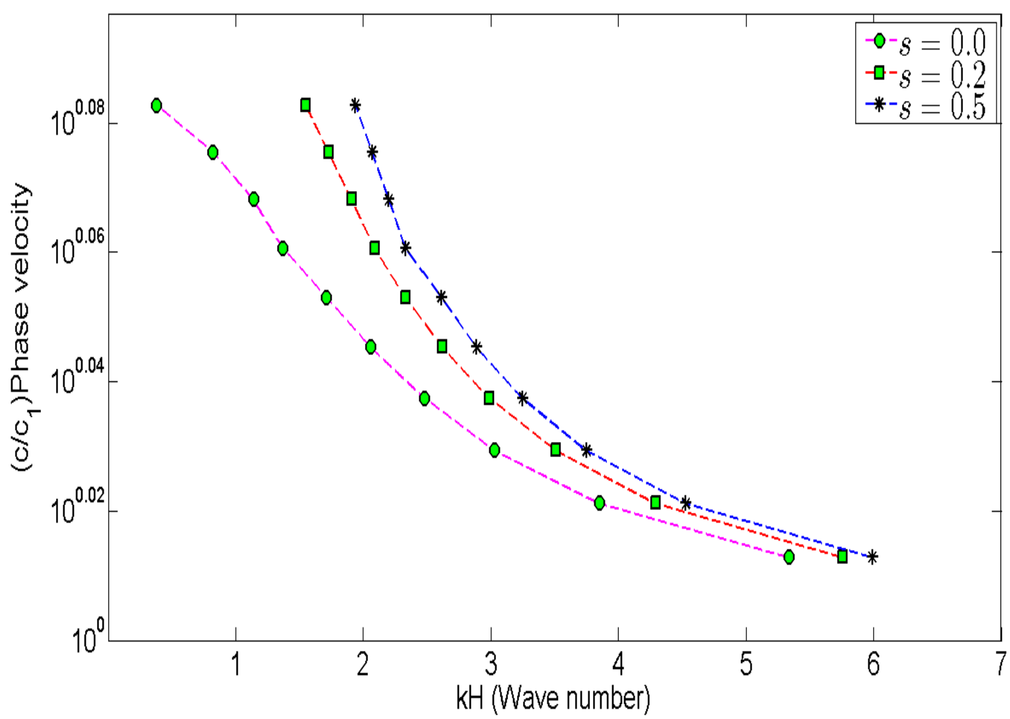

Figure 2.4: Phase velocity against wave number curves for Love waves

CHAPTER 3

LOVE WAVES PROPAGATING IN A HOMOGENEOUS ANISOTROPIC ELASTIC MATERIALS

In this chapter, we study Love waves in a homogeneous anisotropic elastic layer overlying an anisotropic elastic half-space. We derive the dispersion relation for the Love wave propagation in the medium under consideration.

3.1 Formulation of the problem

We consider a homogeneous anisotropic layer of thickness H which overlies an anisotropic elastic half-space. We assume that the upper half space of the anisotropic layer is free and horizontal. We consider a rectangular coordinate system $Oxyz$ in such a way that the z-axis is taken vertically downward the anisotropic half-space and the x-axis is considered in the direction of propagation of wave. In the layer, we have taken ρ_0 and a_{ik} to be the mass density and the elastic moduli at $z = 0$ respectively. Also in the elastic half-space, we have taken ρ_1 and c_{ik} to be the mass density and elastic moduli respectively. Let (u, v, w) be the components of displacement vector along x, y, z directions respectively. Geometric picture of the problem is shown in Fig. (3.1) below.

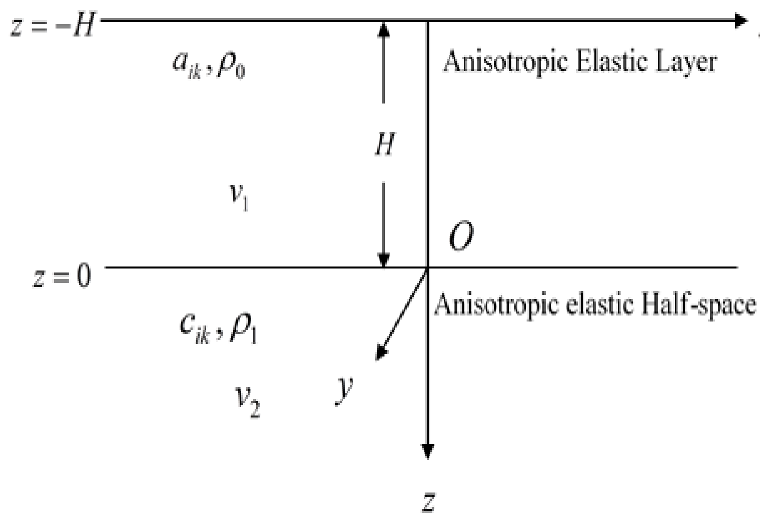

Figure 3.1: Geometry of the problem II.

3.2 Solution of the layer

The equation of motion is given as $\sigma_{ik,i} + X_k = \rho \ddot{u}_k$ for $i, k = 1, 2, 3$. We obtain the equations of motion in upper layer neglecting body forces as given by [12] as

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= \rho_0 \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= \rho_0 \frac{\partial^2 v_1}{\partial t^2} \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= \rho_0 \frac{\partial^2 w_1}{\partial t^2}\end{aligned} \tag{3.1}$$

For Love waves, $u = w = 0$, the body forces as well as the displacement do not depend on y and v is the function of x, z and t. This tells us that two of the three equations of motion given above are identically zero. Hence we have

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = \rho_0 \frac{\partial^2 v_1}{\partial t^2} \tag{3.2}$$

From the generalized Hooke's law and assuming we have a linear strain (infinitesimal strain), the stress-strain equation for the anisotropic layer (medium 1) with the assumptions above give us,

$$\begin{aligned}\sigma_{xx} = \sigma_{xz} = \sigma_{yy} = \sigma_{zz} &= 0 \\ \sigma_{yz} &= a_{44}\frac{\partial v_1}{\partial z} + a_{46}\frac{\partial v_1}{\partial x} \\ \sigma_{xy} &= a_{46}\frac{\partial v_1}{\partial z} + a_{66}\frac{\partial v_1}{\partial x}.\end{aligned} \tag{3.3}$$

Substituting (3.3) into(3.2) we have,

$$\frac{\partial^2 v_1}{\partial z^2} + 2\alpha \frac{\partial^2 v_1}{\partial z \partial x} + \beta \frac{\partial^2 v_1}{\partial x^2} = \frac{\rho_0}{a_{44}} \frac{\partial^2 v_1}{\partial t^2}. \tag{3.4}$$

Where $\alpha = \frac{a_{46}}{a_{44}}$, $\beta = \frac{a_{66}}{a_{44}}$ and $c_1^2 = \frac{a_{66}}{\rho_0}$. Assuming that $v_1(x, z, t) = V_1(z)e^{ik(x-ct)}$.

Putting $v_1(x,z,t)$ into 3.4 , we have

$$\frac{d^2V_1}{dz^2} + 2\alpha\frac{dV_1}{dz} - \beta k^2\left(1 - \frac{c^2}{c_1^2}\right)V_1 = 0 \tag{3.5}$$

The solution of the above differential equation (3.5) is given as

$$V_1(z) = Re^{-ik\xi_1 z} + Se^{ik\xi_2 z}. \tag{3.6}$$

Where $\xi_1 = \alpha + \sqrt{\alpha^2 + \beta\left(\frac{c^2}{c_1^2} - 1\right)}$ and $\xi_2 = -\alpha + \sqrt{\alpha^2 + \beta\left(\frac{c^2}{c_1^2} - 1\right)}$. Hence the displacement and stress component for the anisotropic layer are given as

$$v_1(x,z,t) = \left[Re^{-ik\xi_1 z} + Se^{ik\xi_2 z}\right]e^{ik(x-ct)} \tag{3.7}$$

$$\sigma_{yz} = a_{44}\frac{\partial v_1}{\partial z} + a_{46}\frac{\partial v_1}{\partial x}. \tag{3.8}$$

3.3 Solution in the half-space

For the solution in the anisotropic elastic half-space, the equation of motion for Love waves in the absence of body forces is given by

$$\frac{\partial \eta_{xy}}{\partial x} + \frac{\partial \eta_{yz}}{\partial z} = \rho_1\frac{\partial^2 v_2}{\partial x^2} \tag{3.9}$$

Similarly, the stress-strain relations are given below as,

$$\begin{aligned} \eta_{xx} = \eta_{xz} = \eta_{yy} = \eta_{zz} = 0 \\ \eta_{yz} = c_{44}\frac{\partial v_2}{\partial z} + c_{46}\frac{\partial v_2}{\partial x} \\ \eta_{xy} = c_{46}\frac{\partial v_2}{\partial z} + c_{66}\frac{\partial v_2}{\partial x} \end{aligned} \tag{3.10}$$

Where η_{xy}, η_{xx}, η_{yy}, η_{zz}, η_{yz}, η_{xz}, η_{yx}, η_{zx} and η_{yz} are stresses. Substituting (3.10) into (3.9), we obtain

$$\frac{\partial^2 v_2}{\partial z^2} + 2\gamma\frac{\partial^2 v_2}{\partial z \partial x} + \psi\frac{\partial^2 v_2}{\partial x^2} = \frac{\rho_1}{c_{44}}\frac{\partial^2 v_2}{\partial t^2} \tag{3.11}$$

Where $\gamma = \frac{c_{46}}{c_{44}}$, $\psi = \frac{c_{66}}{c_{44}}$ and $c_2^2 = \frac{c_{66}}{\rho_1}$. Assuming that $v_2(x, z, t) = V_2(z)e^{ik(x-ct)}$. Putting $v_2(x, z, t)$ into (3.11) , we have

$$\frac{d^2 V_2}{dz^2} + 2\gamma\frac{dV_2}{dz} - \psi k^2\Big(1 - \frac{c^2}{c_2^2}\Big)V_2 = 0 \tag{3.12}$$

The solution of the above differential equation (3.12) is given as

$$V_2(z) = Te^{-ik\zeta_1 z} + Ye^{ik\zeta_2 z} \tag{3.13}$$

Where $\zeta_1 = \gamma + \sqrt{\gamma^2 + \psi\Big(\frac{c^2}{c_1^2} - 1\Big)}$ and $\zeta_2 = -\gamma + \sqrt{\gamma^2 + \psi\Big(\frac{c^2}{c_1^2} - 1\Big)}$. Hence the displacement for the lower half-space and stress component for the anisotropic lower elastic half-space are given as

$$v_2(x, z, t) = Te^{-ik\zeta_1 z}e^{ik(x-ct)} \tag{3.14}$$

$$\eta_{yz} = c_{44}\frac{\partial v_2}{\partial z} + c_{46}\frac{\partial v_2}{\partial x} \tag{3.15}$$

.

3.4 Boundary conditions

The appropriate boundary conditions for the propagation of Love wave are given as follows; with the assumption that the anisotropic elastic layer and the anisotropic elatic half space are perfectly bonded.

1. At the interface $z = -H$, $(\sigma_{yz})_{medium\ 1} = 0$ where there is no stress due to free boundary surface.

2. At the interface $z = 0$, the displacement components are continuous that is $v_1 = v_2$.

3. At the interface $z = 0$, the stress is continuous, that is, $(\sigma_{yz})_{medium1} = (\eta_{yz})_{medium\ 2}$.

.

3.5 Dispersion relation

We use the boundary conditions as given in section (3.4), as numbered from (1) to (3) into (3.7), (3.8), (3.14) and (3.15) we have three homogeneous equations given below as

$$R + S - T = 0 \tag{3.16}$$

$$(a_{46} - a_{44}\xi_1)R + (a_{46} + a_{44}\xi_2)S - (c_{46} - c_{44}\zeta_1)T = 0 \tag{3.17}$$

$$(a_{46}e^{ik\xi_1 H} - a_{44}\xi_1 e^{ik\xi_1 H})R + (a_{46}e^{-ik\xi_2 H} + a_{44}\xi_2 e^{-ik\xi_2 H})S = 0 \tag{3.18}$$

We now eliminate R, S and T from the three homogeneous equations and we have

$$\begin{vmatrix} 1 & 1 & -1 \\ (a_{46} - a_{44}\xi_1) & (a_{46} + a_{44}\xi_2) & -(c_{46} - c_{44}\zeta_1) \\ (a_{46}e^{ik\xi_1 H} - a_{44}\xi_1 e^{ik\xi_1 H}) & (a_{46}e^{-ik\xi_2 H} + a_{44}\xi_2 e^{-ik\xi_2 H}) & 0 \end{vmatrix} = 0 \tag{3.19}$$

Simplifying the determinant form (3.19) we have the dispersion relation as

$$tanh\left[ikH\sqrt{\alpha^2 + \beta\left(\frac{c^2}{c_1^2} - 1\right)}\right] = \frac{\theta\sqrt{\alpha^2+\beta\left(\frac{c^2}{c_1^2}-1\right)}\sqrt{\gamma^2+\psi\left(\frac{c^2}{c_2^2}-1\right)}}{\beta\left(\frac{c^2}{c_1^2}-1\right)+\alpha+\sqrt{\alpha^2+\beta\left(\frac{c^2}{c_1^2}-1\right)}-\left[\gamma+\sqrt{\gamma^2+\psi\left(\frac{c^2}{c_2^2}-1\right)}\right]\Phi} \tag{3.20}$$

Where $\theta = \frac{c_{44}}{a_{44}}$ and $\Phi = \frac{c_{46}}{a_{44}}$. Equation (3.20) is the dispersion relation for the Love waves problem under discussion.

3.5.1 Isotropic case

When we assume that the media under consideration is isotropic, we have that $a_{44} = a_{66} = \mu_1$, $a_{46} = 0$, $c_{44} = c_{66} = \mu_2$, $c_{46} = 0$. So this means the dispersion equation (3.20) becomes,

$$tanh\left[ikH\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}\right]=\frac{\mu_2}{\mu_1}\frac{\sqrt{\left(\frac{c^2}{c_2^2}-1\right)}}{\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}}$$

and using the fact that $tanh\left[ikH\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}\right]=itan\left[kH\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}\right]$, we have

$$tan\left[kH\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}\right]=\frac{\mu_2}{\mu_1}\frac{\sqrt{\left(1-\frac{c^2}{c_2^2}\right)}}{\sqrt{\left(\frac{c^2}{c_1^2}-1\right)}} \tag{3.21}$$

Equation (3.21) corresponds to the classical Love wave dispersion as given by [1]

3.6 Numerical results and Discussion

In order to show the dependence of phase velocity on wave number, we have taken data for the anisotropic layer from [73] as follows $a_{44}=8.30Gpa$, $a_{46}=0.06Gpa$, $a_{66}=7.77Gpa$ and $\rho_0=2216kg/m^3$. For anisotropic elastic half-space we have taken data from [54] as follows; $c_{44}=25.97Gpa$, $c_{46}=0.43Gpa$, $c_{66}=37.82Gpa$ and $\rho_1=2727kg/m^3$. Using the above data and the dispersion relation in equation (3.20), we have the table below and plot graphs of phase velocity against wave number as below.

In the figures below curves are plotted separately for both real and imaginary parts of phase velocity against wave number. Fig. (3.2) depicts a graph of dimensionless phase velocity real (c/c_1) against the dimensionless wave number kH for value of thickness of layer $(H=10Km)$. The curves reveal that the dimensionless phase velocity decreases for increasing dimensionless wave number for $H = 10Km$. It is also seen that as we increase the thickness of layer H, the magnitude of phase velocity decreases for all values of kH . Fig. (3.3) depicts a graph of phase velocity imaginary (c/c_1) against wave number. The curves reveal that the dimensionless phase velocity decreases for increasing dimensionless wave number for $H = 10Km$. Similarly, as we increase the thickness of layer, the magnitude of phase velocity decreases for all values of kH. We also observe from both figures that, the rate at which the phase velocity decreases in the real case is a bit greater than for the imaginary phase velocity.

Phase velocity	$k(Real)$	$k(Imaginary)$
$Re(\frac{c}{c_1})$ and $Im(\frac{c}{c_1})$		
1.03	0.622227	0.0288563
1.05	0.473087	0.0170734
1.07	0.393088	0.0120269
1.09	0.341169	0.00922475
1.11	0.303906	0.00744247
1.13	0.275432	0.0062092
1.15	0.252719	0.00530523
1.17	0.234023	0.00461428
1.19	0.218261	0.004069
1.21	0.204719	0.0032773

Table 3.1: This is a table that shows the values of both real and imaginary dimensionless phase velocities and their respective dimensionless wave numbers at $H = 10Km$.

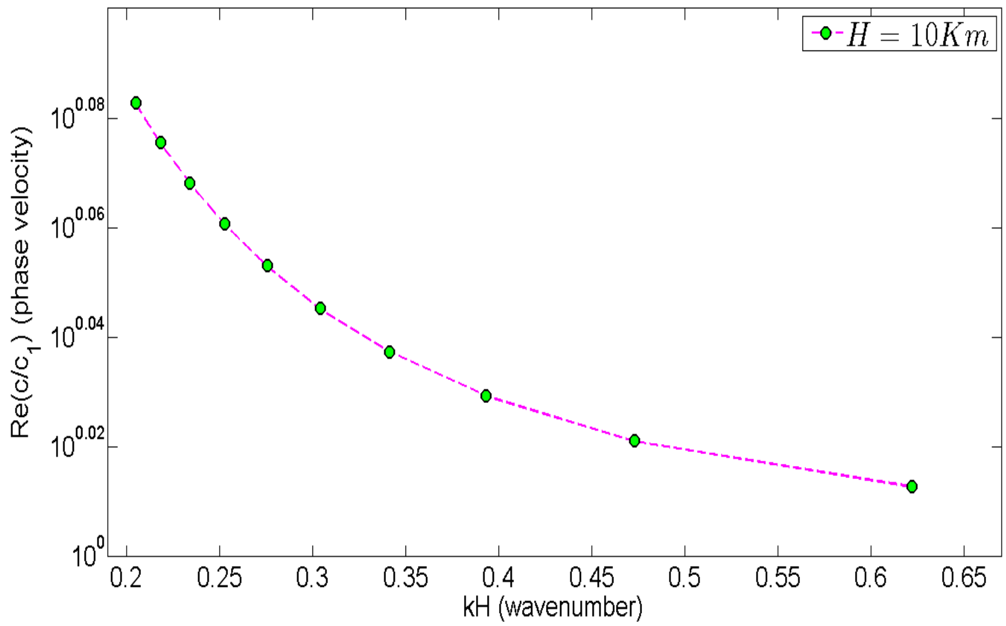

Figure 3.2: Dimensionless phase velocity curve real (c/c_1) against dimensionless wave number kH for $H = 10km$

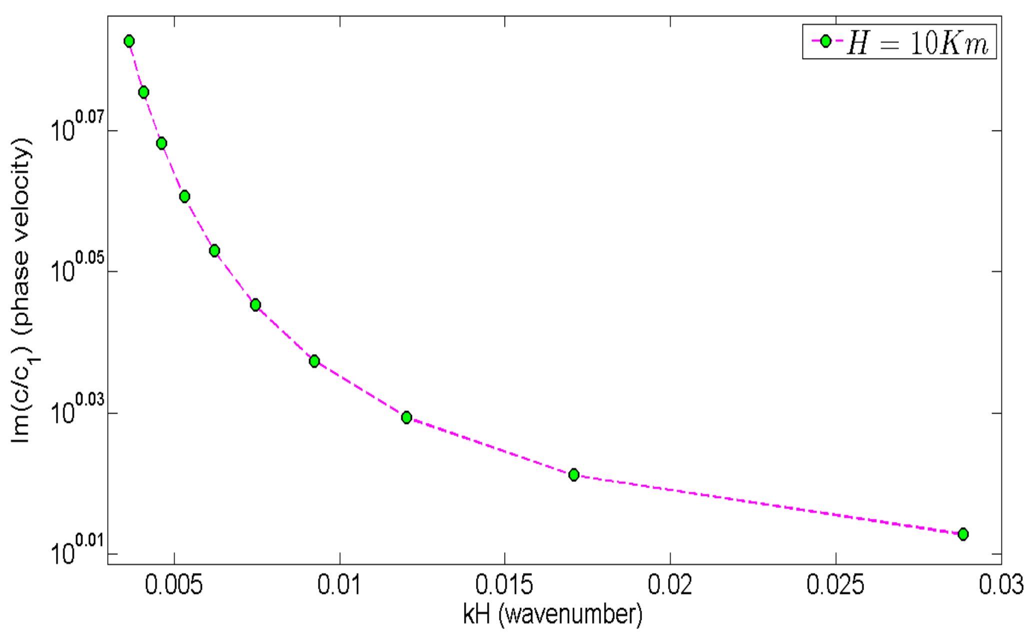

Figure 3.3: Dimensionless Phase velocity curve imaginary (c/c_1) against dimensionless wave number kH for $H = 10km$

CHAPTER 4

LOVE WAVES PROPAGATING IN AN INHOMOGENEOUS ANISOTROPIC ELASTIC MATERIALS

In this chapter, we consider propagation of Love waves in an inhomogeneous anisotropic layer lying over an anisotropic elastic half-space. We solve the problem and obtain the dispersion relation in the determinant form using the perturbation series approach.

4.1 Statement of the problem

We consider an inhomogeneous anisotropic layer of thickness H overlying an anisotropic elastic half-space. We assume that the overlying anisotropic layer is free and horizontal. We consider the rectangular coordinate system $Oxyz$ in such a way that the z-axis is taken vertically downward the anisotropic half-space and the x-axis is considered in the direction of propagation of wave. In this section, the inhomogeneities in the upper layer have been taken in the form $\rho_1 = \rho_0(1 + \varepsilon_2 z)$ and $C_{ik} = c_{ik}(1 + \varepsilon_1 z)$, where ρ_0 and c_{ik} denote the mass density and elastic moduli at $z = 0$, respectively and ε_1 and ε_2 having dimensions that are inverse of length (L^{-1}). We let ρ_2 and a_{ik} represent the density and the elastic moduli in the half-space respectively. Let (u, v, w) are the components of displacement vector along x, y, z directions respectively. Geometric picture of the problem is shown in Fig. 4.1.

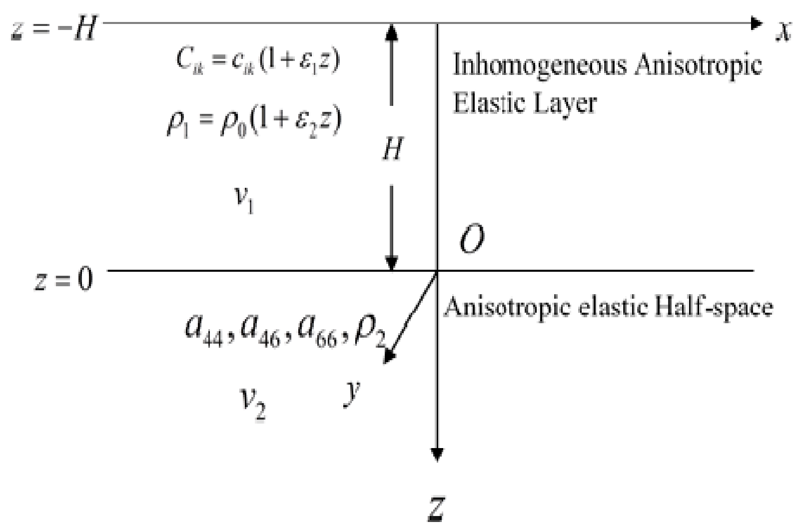

Figure 4.1: Geometry of the problem III.

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4.2 Solution of the layer

We obtain the equations of motion in upper layer neglecting body forces as given by [12] as

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= \rho_1 \frac{\partial^2 u_1}{\partial t^2} \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= \rho_1 \frac{\partial^2 v_1}{\partial t^2} \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= \rho_1 \frac{\partial^2 w_1}{\partial t^2}
\end{aligned} \tag{4.1}$$

We know from the generalized Hooke's law that there exists a linear relationship between the stress and strain tensors given below as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix}$$

This gives,

$$\sigma_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + 2C_{14}\varepsilon_{yz} + 2C_{15}\varepsilon_{xz} + 2C_{16}\varepsilon_{xy}$$

$$\sigma_{yy} = C_{12}\varepsilon_{xx} + C_{22}\varepsilon_{yy} + C_{23}\varepsilon_{zz} + 2C_{24}\varepsilon_{yz} + 2C_{25}\varepsilon_{xz} + 2C_{26}\varepsilon_{xy}$$

$$\sigma_{zz} = C_{13}\varepsilon_{xx} + C_{23}\varepsilon_{yy} + C_{33}\varepsilon_{zz} + 2C_{34}\varepsilon_{yz} + 2C_{35}\varepsilon_{xz} + 2C_{36}\varepsilon_{xy}$$

$$\sigma_{xy} = C_{14}\varepsilon_{xx} + C_{24}\varepsilon_{yy} + C_{34}\varepsilon_{zz} + 2C_{44}\varepsilon_{yz} + 2C_{45}\varepsilon_{xz} + 2C_{46}\varepsilon_{xy}$$

$$\sigma_{yz} = C_{15}\varepsilon_{xx} + C_{25}\varepsilon_{yy} + C_{35}\varepsilon_{zz} + 2C_{45}\varepsilon_{yz} + 2C_{55}\varepsilon_{xz} + 2C_{56}\varepsilon_{xy}$$

$$\sigma_{zx} = C_{16}\varepsilon_{xx} + C_{26}\varepsilon_{yy} + C_{36}\varepsilon_{zz} + 2C_{46}\varepsilon_{yz} + 2C_{56}\varepsilon_{xz} + 2C_{66}\varepsilon_{xy}$$

The dynamical equation of motion for Love waves propagating along x-axis in non-homogeneous, anisotropic elastic medium is given by [43].

$$u_1 = 0, \quad w_1 = 0, \quad v_1 = (x, z, t) \tag{4.2}$$

This leads to

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = \rho_1 \frac{\partial^2 v_1}{\partial t^2} \tag{4.3}$$

Where ρ_1 is the density of the layer. The strain-displacement relations are given as

$$\begin{gathered}
\varepsilon_{xx} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial x}\right) = \frac{\partial u_1}{\partial x}, \;\; \varepsilon_{yy} = \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_1}{\partial y}\right) = \frac{\partial v_1}{\partial y}, \\
\varepsilon_{zz} = \frac{1}{2}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_1}{\partial z}\right) = \frac{\partial w_1}{\partial z}, \;\; \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y}\right), \\
\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x}\right), \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)
\end{gathered}$$

The stress-strain relations for an inhomogeneous anisotropic layer with the assumptions in equation (4.2) are the form

$$\begin{aligned} \sigma_{xx} = \sigma_{xz} = \sigma_{yy} = \sigma_{zz} = 0 \\ \sigma_{yz} = c_{44}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial z} + c_{46}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial x} \\ \sigma_{xy} = c_{46}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial z} + c_{66}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial x} \end{aligned} \tag{4.4}$$

Substituting equation (4.4) into equation (4.3) gives,

$$\frac{\partial}{\partial x}\Big[c_{46}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial z} + c_{66}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial x}\Big] + \frac{\partial}{\partial z}\Big[c_{44}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial z} + c_{46}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial x}\Big] = \rho_1 \frac{\partial^2 v_1}{\partial t^2}$$

$$\begin{gathered} c_{46}(1+\varepsilon_1 z)\frac{\partial^2 v_1}{\partial x \partial z} + c_{66}(1+\varepsilon_1 z)\frac{\partial^2 v_1}{\partial x^2} + c_{44}(1+\varepsilon_1 z)\frac{\partial^2 v_1}{\partial z^2} + c_{44}\varepsilon_1 \frac{\partial v_1}{\partial z} + \\ c_{46}\varepsilon_1 \frac{\partial v_1}{\partial x} c_{46}(1+\varepsilon_1 z)\frac{\partial^2 v_1}{\partial z \partial x} = \rho_0(1+\varepsilon_2 z)\frac{\partial^2 v_1}{\partial t^2} \end{gathered}$$

dividing through by $(1+\varepsilon_1 z)$ we obtain,

$$c_{44}\frac{\partial^2 v_1}{\partial z^2} + 2c_{46}\frac{\partial^2 v_1}{\partial x \partial z} + c_{66}\frac{\partial^2 v_1}{\partial x^2} + c_{44}\frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial z} + c_{46}\frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial x} = \rho_0 \frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\frac{\partial^2 v_1}{\partial t^2} \tag{4.5}$$

Dividing equation (4.5) by c_{44} we have,

$$\frac{\partial^2 v_1}{\partial z^2} + \beta \frac{\partial^2 v_1}{\partial x^2} + 2\alpha \frac{\partial^2 v_1}{\partial x \partial z} + \frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial z} + \alpha \frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial x} = \frac{\rho_0}{c_{44}}\frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\frac{\partial^2 v_1}{\partial t^2} \tag{4.6}$$

Where $\frac{c_{46}}{c_{44}} = \alpha$, $\frac{c_{66}}{c_{44}} = \beta$. We find the solution of equation (4.6) in the form $v_1(x,z,t) = \phi(z)e^{ik(x-ct)}$. This gives

$$\frac{d^2\phi(z)}{dz^2} + \Big(2i\alpha k + \frac{\varepsilon_1}{(1+\varepsilon_1 z)}\Big)\frac{d\phi(z)}{dz} - k^2\beta\Big[1 - \frac{i\gamma\varepsilon_1}{k(1+\varepsilon_1 z)} - \frac{c^2}{c_1^2}\frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\Big]\phi(z) = 0 \tag{4.7}$$

where $\gamma = \dfrac{c_{46}}{c_{66}}$, and $c_1^2 = \dfrac{c_{66}}{\rho_0}$. We now normalize to eliminate the term $\dfrac{d\phi(z)}{dz}$ by substituting $\phi(z) = \dfrac{e^{-ik\alpha z}\psi(z)}{\sqrt{1+\varepsilon_1 z}}$. Differentiating $\phi(z) = \dfrac{e^{-ik\alpha z}\psi(z)}{\sqrt{1+\varepsilon_1 z}}$ with respect to z we have

$$\frac{d\phi}{dz} = \frac{-ik\alpha e^{-ik\alpha z}\psi(z)}{(1+\varepsilon_1 z)^{1/2}} + \frac{e^{-ik\alpha z}\psi\prime(z)}{(1+\varepsilon_1 z)^{1/2}} - \frac{1}{2}\frac{\varepsilon_1 e^{-ik\alpha z}\psi(z)}{(1+\varepsilon_1 z)^{3/2}}$$

Differentiating $\phi(z)$ for the second time with respect to z, gives

$$\frac{d^2\phi}{dz^2} = \frac{\alpha^2 k^2 e^{-ik\alpha z}\psi(z)}{(1+\varepsilon_1 z)^{1/2}} - \frac{i\alpha k e^{-ik\alpha z}\psi'(z)}{(1+\varepsilon_1 z)^{1/2}} + \frac{ik\alpha e^{-ik\alpha z}\psi(z)\varepsilon_1}{(1+\varepsilon_1 z)^{3/2}} + \frac{e^{-ik\alpha z}\psi''(z)}{(1+\varepsilon_1 z)^{1/2}} - \frac{1}{2}\frac{\varepsilon_1 e^{-ik\alpha z}\psi(z)}{(1+\varepsilon_1 z)^{3/2}} - \frac{1}{2}\frac{\varepsilon_1 e^{-ik\alpha z}\psi'(z)}{(1+\varepsilon_1 z)^{3/2}} + \frac{3}{4}\frac{(\varepsilon_1)^2\varepsilon_1 e^{-ik\alpha z}\psi(z)}{(1+\varepsilon_1 z)^{5/2}}$$

Substituting the first and second derivatives of ϕ into equation (4.7) we have

$$\psi''(z) + \left[\frac{(\varepsilon_1)^2}{4(1+\varepsilon_1 z)^2} - k^2\left[(\beta-\alpha^2) - \frac{i\alpha\varepsilon_1(\beta\gamma_1-\alpha)}{k(1+\varepsilon_1 z)} - \frac{\beta c^2}{c_1^2}\frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\right]\right]\psi(z) = 0.$$

$$\frac{d^2\psi(z)}{dz^2} + \left[\frac{(\varepsilon_1)^2}{4(1+\varepsilon_1 z)^2} - k^2\left[(\beta-\alpha^2) - \frac{i\alpha\varepsilon_1(\beta\gamma_1-\alpha)}{k(1+\varepsilon_1 z)} - \frac{\beta c^2}{c_1^2}\frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\right]\right]\psi(z) = 0 \tag{4.8}$$

We now introduce $\zeta = \left[(\beta-\alpha^2) - \dfrac{i\alpha\varepsilon_1(\beta\gamma_1-\alpha)}{k(1+\varepsilon_1 z)} - \dfrac{\beta c^2}{c_1^2}\dfrac{\varepsilon_2}{\varepsilon_1}\right]^{1/2}$ and $\tau = \dfrac{2\zeta k(1+\varepsilon_1 z)}{\varepsilon_1}$ in equation (4.8). We thus obtain

$$\frac{d^2\psi(\tau)}{dz^2} + \left(\frac{f}{2\tau} + \frac{1}{4\tau^2} - \frac{1}{4}\right)\psi(\tau) = 0. \tag{4.9}$$

Where $\omega = kc$ and $f = \dfrac{-\beta\omega^2\varepsilon_2 + \beta\omega^2\varepsilon_1}{c_1^2\varepsilon_1^2\zeta k}$. From equation (4.9) we notice that it is similar to the Whittaker equation given by Whittaker and Watson (1991) [75]. The solution to (4.9) above can be written as,

$$\psi(\tau) = S_1 W_{\frac{s}{2},0}(\tau) + S_2 W_{-\frac{s}{2},0}(-\tau) \tag{4.10}$$

where S_1 and S_2 are arbitrary constants and $W_{\frac{s}{2},0}(\tau)$ and $W_{-\frac{s}{2},0}(-\tau)$ are given as

$$W_{\frac{s}{2},0}(\tau) = e^{-\frac{\zeta k(1+\varepsilon_1 z)}{\varepsilon_1}}\left(\frac{2\zeta k(1+\varepsilon_1 z)}{\varepsilon_1}\right)^{\frac{f}{2}}\left(1 - \frac{\left(\frac{f-1}{2}\right)^2\varepsilon_1}{2\zeta k(1+\varepsilon_1 z)}\right)$$

$$W_{\frac{s}{2},0}(-\tau) = e^{\frac{\zeta k(1+\varepsilon_1 z)}{\varepsilon_1}}\left(-\frac{2\zeta k(1+\varepsilon_1 z)}{\varepsilon_1}\right)^{-\frac{f}{2}}\left(1 - \frac{\left(\frac{f-1}{2}\right)^2\varepsilon_1}{2\zeta k(1+\varepsilon_1 z)}\right)$$

Hence the mechanical displacement for the upper layer is the solution in equation (4.10) and it is denoted as

$$v_1(x,z,t) = \phi(z)e^{ik(x-ct)} = \left(\frac{S_1 W_{\frac{s}{2},0}(\tau)e^{-ik\alpha z} + S_2 W_{-\frac{s}{2},0}(-\tau)e^{-ik\alpha z}}{(1+\varepsilon_1)^{1/2}}\right)e^{ik(x-ct)} \tag{4.11}$$

$$(\sigma_{23})_{medium1} = c_{44}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial z} + c_{46}(1+\varepsilon_1 z)\frac{\partial v_1}{\partial x} \tag{4.12}$$

4.3 Solution in the half-space

We consider the equation of motion for half-space given by [12] as

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= \rho_2 \frac{\partial^2 u_2}{\partial t^2}\\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= \rho_2 \frac{\partial^2 v_2}{\partial t^2}\\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} &= \rho_2 \frac{\partial^2 w_2}{\partial t^2}
\end{aligned} \tag{4.13}$$

Here τ_{xx}, τ_{xy}, τ_{xz}, τ_{yy}, τ_{yz}, τ_{zx}, τ_{zy} and τ_{zz} are stress components. u_2, v_2 and w_2 are the components of the displacement vector in the upper layer, ρ_2 is the density of the elastic medium. Again using the assumptions that

$$u_2 = 0, w_2 = 0, v_2 = (x, z, t) \quad \text{and} \quad \frac{\partial}{\partial y} = 0 \tag{4.14}$$

given by Love (1911) [43], we obtain an equation of motion for Love waves without body force as,

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = \rho_2 \frac{\partial^2 v_2}{\partial t^2} \tag{4.15}$$

The stress-strain relations for a general anisotropic elastic medium are given as

$$\tau_{xx} = a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{yy} + a_{13}\varepsilon_{zz} + 2a_{14}\varepsilon_{yz} + 2a_{15}\varepsilon_{xz} + 2a_{16}\varepsilon_{xy}$$

$$\tau_{yy} = a_{12}\varepsilon_{xx} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{zz} + 2a_{24}\varepsilon_{yz} + 2a_{25}\varepsilon_{xz} + 2a_{26}\varepsilon_{xy}$$

$$\tau_{zz} = a_{13}\varepsilon_{xx} + a_{23}\varepsilon_{yy} + a_{33}\varepsilon_{zz} + 2a_{34}\varepsilon_{yz} + 2a_{35}\varepsilon_{xz} + 2a_{36}\varepsilon_{xy}$$

$$\tau_{xy} = a_{14}\varepsilon_{xx} + a_{24}\varepsilon_{yy} + a_{34}\varepsilon_{zz} + 2a_{44}\varepsilon_{yz} + 2a_{45}\varepsilon_{xz} + 2a_{46}\varepsilon_{xy}$$

$$\tau_{yz} = a_{15}\varepsilon_{xx} + a_{25}\varepsilon_{yy} + a_{35}\varepsilon_{zz} + 2a_{45}\varepsilon_{yz} + 2a_{55}\varepsilon_{xz} + 2a_{56}\varepsilon_{xy}$$

$$\tau_{zx} = a_{16}\varepsilon_{xx} + a_{26}\varepsilon_{22} + a_{36}\varepsilon_{33} + 2a_{46}\varepsilon_{23} + 2a_{56}\varepsilon_{xz} + 2a_{66}\varepsilon_{xy}$$

We obtain the stress-strain relations for a general anisotropic elastic medium as already established in section (4.2) as

$$\begin{aligned} \tau_{xx} = \tau_{xz} = \tau_{yy} = \tau_{zz} = 0 \\ \tau_{yz} = a_{44}\frac{\partial v_2}{\partial z} + a_{46}\frac{\partial v_2}{\partial x} \\ \tau_{xy} = a_{46}\frac{\partial v_2}{\partial z} + a_{66}\frac{\partial v_2}{\partial x} \end{aligned} \tag{4.16}$$

We substitute equation (4.16) into equation (4.15) and we have as follows

$$\frac{\partial}{\partial x}\left[a_{46}\frac{\partial v_2}{\partial z}+a_{66}\frac{\partial v_1}{\partial x}\right]+\frac{\partial}{\partial z}\left[a_{44}\frac{\partial v_2}{\partial z}+a_{46}\frac{\partial v_2}{\partial x}\right]=\rho_2\frac{\partial^2 v_2}{\partial t^2}$$

$$a_{44}\frac{\partial^2 v_2}{\partial z^2}+2a_{46}\frac{\partial^2 v_2}{\partial x\partial z}+a_{66}\frac{\partial^2 v_2}{\partial x^2}=\rho_2\frac{\partial^2 v_2}{\partial t^2}$$

Dividing throughout by a_{44} we obtain,

$$\frac{\partial^2 v_2}{\partial z^2}+2\frac{a_{46}}{c_{44}}\frac{\partial^2 v_2}{\partial x\partial z}+\frac{a_{66}}{a_{44}}\frac{\partial^2 v_2}{\partial x^2}=\frac{\rho_2}{a_{44}}\frac{\partial^2 v_2}{\partial t^2}.$$

where $b=\dfrac{a_{66}}{a_{44}}$, $q=\dfrac{a_{46}}{a_{44}}$ and $c_2^2=\dfrac{a_{66}}{\rho_2}$

$$\frac{\partial^2 v_2}{\partial z^2}+2q\frac{\partial^2 v_2}{\partial x\partial z}+b\frac{\partial^2 v_2}{\partial x^2}=\frac{\rho_2}{a_{44}}\frac{\partial^2 v_2}{\partial t^2}$$

Now let $v_2=V_2(z)e^{ik(x-ct)}$, which gives us;

$$\frac{d^2V_2}{dz^2}+2qik\frac{dV_2}{dz}-bk^2\left(1-\frac{c^2}{c_2^2}\right)V_2=0$$

Where $c_2^2=\dfrac{a_{66}}{\rho_2}$. Solving the second order differential equation we have

$$V_2(z)=B_1e^{-ik\lambda_1}+B_2e^{ik\lambda_2 z}$$

where $\lambda_1=q+\sqrt{q^2+b\Big(\frac{c^2}{c_2^2}-1\Big)}$ and $\lambda_2=-q+\sqrt{q^2+b\Big(\frac{c^2}{c_2^2}-1\Big)}$

Hence the displacement for the lower half-space as well as the stress in the anisotropic elastics half space are given as

$$v_2(x,z,t)=B_1e^{-ik\lambda_1 z}e^{ik(x-ct)} \tag{4.17}$$

$$(\tau_{yz})_{medium2}=a_{44}\frac{\partial v_2}{\partial z}+a_{46}\frac{\partial v_1}{\partial x}. \tag{4.18}$$

4.4 Boundary conditions

The appropriate boundary conditions for the propagation of Love wave are given as follows with the assumption that the inhomogeneous anisotropic elastic layer and the anisotropic elatic half space are perfectly bonded.

1. At the free surface $z = -H$, $(\sigma_{yz})_{medium1} = 0$ as the normal component of stress vanishes.

2. At the interface $z = 0$, the displacement components are continuous that is $v_1(z) = v_2(z)$.

3. At the interface $z = 0$, the stress is continuous, that is, $(\sigma_{yz})_{medium1} = (\tau_{yz})_{medium2}$.

4.5 Dispersion relation

We use the boundary conditions as given in Section 4.4 and equations (4.11), (4.12), (4.17) and (4.18), to obtain respectively,

$$(J_1 J_2)S_1 + (J_4 J_5)S_2 - B_1 = 0 \tag{4.19}$$

$$J_9 S_1 + J_{10} S_2 + ik(c_{46} - \lambda_1 c_{44})B_1 = 0 \tag{4.20}$$

$$J_{11} S_1 + J_{12} S_2 = 0 \tag{4.21}$$

Where $J_9 = (J_1J_2J_7c_{44} + J_1J_3J_8c_{44} + ikJ_1J_2c_{46})$, $J_{10} = (J_4J_5J_7c_{44} + J_4J_6J_8c_{44} + ikJ_4J_5c_{46})$, $J_{11} = (J'_1J'_2J'_7c_{44} + J'_1J'_3J'_8c_{44} + ikJ'_1J'_2c_{46})$ and $J_{12} = (J'_4J'_5J'_7c_{44} + J'_4J'_6J'_8c_{44} + ikJ'_4J'_5c_{46})$. Eliminating S_1, S_2 and B_1 from eq. (4.21), (4.22) we have,

$$\begin{vmatrix} J_1J_2 & J_4J_5 & -1 \\ J_9 & J_{10} & ik(c_{46} - \lambda_1 c_{44}) \\ J_{11} & J_{12} & 0 \end{vmatrix} = 0 \tag{4.22}$$

Where J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8, J'_1, J'_2, J'_3, J'_4, J'_5, J'_6, J'_7 and J'_8 are given in the Appendix I. The determinant form in Equation (4.22) is the dispersion relation.

4.6 Alternative solution in the layer

We find the solution for the layer of the above problem in Section 4.1 formulation by using the perturbation series method. We follow similar process as in Section 4.3.

$$\frac{\partial^2 v_1}{\partial z^2} + \frac{c_{66}}{c_{44}}\frac{\partial^2 v_1}{\partial x^2} + 2\frac{c_{46}}{c_{44}}\frac{\partial^2 v_1}{\partial x \partial z} + \frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial z} + \frac{c_{46}}{c_{44}}\frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial x} = \frac{\rho_0}{c_{44}}\frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\frac{\partial^2 v_1}{\partial t^2} \tag{4.23}$$

again $\alpha = \dfrac{c_{46}}{c_{44}}$ and $\beta = \dfrac{c_{66}}{c_{44}}$.

$$\frac{\partial^2 v_1}{\partial z^2} + \beta\frac{\partial^2 v_1}{\partial x^2} + 2\alpha\frac{\partial^2 v_1}{\partial x \partial z} + \frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial z} + \alpha\frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial x} = \frac{\rho_0}{c_{44}}\frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\frac{\partial^2 v_1}{\partial t^2} \tag{4.24}$$

Here the time harmonic variation is taken as $e^{i\omega t}$ and be suppressed throughout.

So we have,

$$\frac{\partial^2 v_1}{\partial z^2}+\beta\frac{\partial^2 v_1}{\partial x^2}+2\alpha\frac{\partial^2 v_1}{\partial x\partial z}+\frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial z}+\alpha\frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial x}+\frac{\rho_0}{c_{44}}\frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\omega^2 v_1=0 \tag{4.25}$$

Taking inhomogeneous terms to the right hand side we get

$$\frac{\partial^2 v_1}{\partial z^2}+\beta\frac{\partial^2 v_1}{\partial x^2}+2\alpha\frac{\partial^2 v_1}{\partial x\partial z}=-\frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial z}-\alpha\frac{\varepsilon_1}{(1+\varepsilon_1 z)}\frac{\partial v_1}{\partial x}-\frac{\rho_0}{c_{44}}\frac{(1+\varepsilon_2 z)}{(1+\varepsilon_1 z)}\omega^2 v_1 \tag{4.26}$$

From binomial series expansion,

$$\frac{1}{(1+\varepsilon_1 z)}=(1+\varepsilon_1 z)^{-1}=(1-\varepsilon_1 z+\varepsilon_1^2 z^2-...) \tag{4.27}$$

Substituting equation (4.27) into equation (4.26) we have,

$$\frac{\partial^2 v_1}{\partial z^2}+\beta\frac{\partial^2 v_1}{\partial x^2}+2\alpha\frac{\partial^2 v_1}{\partial x\partial z}=-\varepsilon_1(1-\varepsilon_1 z+\varepsilon_1^2 z^2-...)\frac{\partial v_1}{\partial z}-\alpha\varepsilon_1(1-\varepsilon_1 z+\varepsilon_1^2 z^2-...)\frac{\partial v_1}{\partial x}-\frac{\rho_0}{c_{44}}(1+\varepsilon_2 z)(1-\varepsilon_1 z+\varepsilon_1^2 z^2-...)\omega^2 v_1$$

Let $\varepsilon_1=b_1\varepsilon$ and $\varepsilon_2=b_2\varepsilon$ such that $\varepsilon=max\Big[\varepsilon_1,\varepsilon_2\Big]$ so that

$$\frac{\partial^2 v_1}{\partial z^2}+\beta\frac{\partial^2 v_1}{\partial x^2}+2\alpha\frac{\partial^2 v_1}{\partial x\partial z}=(-b_1\varepsilon+b_1^2\varepsilon^2 z+...)\frac{\partial v_1}{\partial z}+(-\alpha b_1\varepsilon+\alpha b_1^2\varepsilon^2 z+...)\frac{\partial v_1}{\partial x}+\frac{\rho_0}{c_{44}}\omega^2(-1+b_1\varepsilon z-b_1^2\varepsilon^2 z-b_2\varepsilon z+b_1 b_2\varepsilon^2 z-...)v_1$$

$$\frac{\partial^2 v_1}{\partial z^2}+\beta\frac{\partial^2 v_1}{\partial x^2}+2\alpha\frac{\partial^2 v_1}{\partial x\partial z}+\frac{\rho_0}{c_{44}}\omega^2 v_1=-b_1\varepsilon\frac{\partial v_1}{\partial z}-\alpha b_1\varepsilon\frac{\partial v_1}{\partial x}+\alpha b_1^2\varepsilon^2 z\frac{\partial v_1}{\partial x}+\frac{\rho_0}{c_{44}}\omega^2 b_1\varepsilon z v_1-\frac{\rho_0}{c_{44}}\omega^2 b_1^2\varepsilon^2 z v_1-\frac{\rho_0}{c_{44}}\omega^2 b_2\varepsilon z v_1+\frac{\rho_0}{c_{44}}\omega^2 b_1 b_2\varepsilon^2 z v_1+... \tag{4.28}$$

Now employing the perturbation series expansion method, we let

$$v_1 = v_1^{(0)} + \varepsilon v_2^{(1)} + \varepsilon^2 v_1^{(2)} + ... \tag{4.29}$$

Substituting equation (4.29) into equation (4.28) we have

$$\begin{aligned} &\frac{\partial^2}{\partial z^2}[v_1^{(0)} + \varepsilon v_2^{(1)} + ...] + \beta\frac{\partial^2}{\partial x^2}[v_1^{(0)} + \varepsilon v_2^{(1)} + ...] + 2\alpha\frac{\partial^2}{\partial x \partial z}[v_1^{(0)} + \\ &\varepsilon v_2^{(1)} + ...] + \frac{\rho_0}{c_{44}}\omega^2[v_1^{(0)} + \varepsilon v_2^{(1)} + ...] = -b_1\varepsilon\frac{\partial v_1}{\partial z} - \alpha b_1 \varepsilon \frac{\partial v_1}{\partial x} + \\ &\alpha b_1^2 \varepsilon^2 z \frac{\partial v_1}{\partial x} + \frac{\rho_0}{c_{44}}\omega^2 b_1 \varepsilon z[v_1^{(0)} + \varepsilon v_2^{(1)} + ...] - \frac{\rho_0}{c_{44}}\omega^2 b_1^2 \varepsilon^2 z[v_1^{(0)} + \varepsilon v_2^{(1)} + ...] - \\ &\frac{\rho_0}{c_{44}}\omega^2 b_2 \varepsilon z[v_1^{(0)} + \varepsilon v_2^{(1)} + ...] + \frac{\rho_0}{c_{44}}\omega^2 b_1 b_2 \varepsilon^2 z[v_1^{(0)} + \varepsilon v_2^{(1)} + ...] + ... \end{aligned} \tag{4.30}$$

Expanding equation (4.30) and comparing coefficients of ε and neglecting higher order terms ε we have

$$\frac{\partial^2 v_1^{(0)}}{\partial z^2} + 2\alpha\frac{\partial^2 v_1^{(0)}}{\partial x \partial z} + \beta\frac{\partial^2 v_1^{(0)}}{\partial x^2} + \frac{\rho_0}{c_{44}}\omega^2 v_1^{(0)} = 0 \tag{4.31}$$

$$\frac{\partial^2 v_1^{(1)}}{\partial z^2} + 2\alpha\frac{\partial^2 v_1^{(1)}}{\partial x \partial z} + \beta\frac{\partial^2 v_1^{(1)}}{\partial x^2} + \frac{\rho_0}{c_{44}}\omega^2 v_1^{(1)} = -b_1\frac{\partial v_1^{(0)}}{\partial z} - \alpha b_1\frac{\partial v_1^{(0)}}{\partial x} + \frac{\rho_0}{c_{44}} z\omega^2 b_1 v_1^{(0)} - \frac{\rho_0}{c_{44}} z\omega^2 b_2 v_1^{(0)} \tag{4.32}$$

We solve equations (4.31) and (4.32) by using the Fourier transform pair with respect to x as,

$$V_1^{(j)}(\xi,z)=\int_{-\infty}^{\infty}v_1^{(j)}(x,z)e^{i\xi x}dx$$

$$v_1^{(j)}(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}V_1^{(j)}(\xi,z)e^{-i\xi x}d\xi \tag{4.33}$$

$$j=0,1$$

Equation (4.31) thus becomes,

$$\frac{d^2V_1^{(0)}}{dz^2}+2i\alpha\xi\frac{dV_1^{(0)}}{dz}-\beta\xi^2\Big(1-\frac{c^2}{c_1^2}\Big)V_1^{(0)}=0 \tag{4.34}$$

Where $\omega=kc$ and $c_1^2=\dfrac{\xi^2 c_{66}}{k^2\rho_0}$. We solve the second order ordinary differential equation (4.34) to obtain

$$V_1^{(0)}(z)=A_1e^{-i\xi s_1 z}+A_2e^{i\xi s_2 z} \tag{4.35}$$

Where $s_1=\alpha+\sqrt{\alpha^2+\beta\big(\frac{c^2}{c_1^2}-1\big)}$ and $s_2=-\alpha+\sqrt{\alpha^2+\beta\big(\frac{c^2}{c_1^2}-1\big)}$

We find the Fourier inverse transform as

$$v_1^{(0)}(x,z)=\tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty}V_1^{(0)}(\xi,z)e^{-i\xi x}d\xi$$

$$v_1^{(0)}(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[A_1e^{-i\xi s_1 z}+A_2e^{i\xi s_2 z}]e^{-i\xi x}d\xi$$

$$v_1^{(0)}(x,z)=-\frac{A_1}{2\pi i}\Big[\frac{1}{(s_1z+x)}\Big]+\frac{A_2}{2\pi i}\Big[\frac{1}{(s_2z-x)}\Big] \tag{4.36}$$

Here we assumed x is large and that both $(s_2 z - x)$ and $(s_1 z + x)$ are real and greater than zero. Next we take Fourier transform of equation (4.32) with respect to x as below

$$\frac{d^2 V_1^{(1)}}{dz^2} + 2i\alpha\xi \frac{dV_1^{(1)}}{dz} - \beta\xi^2 V_1^{(1)} + \frac{\rho_0}{c_{44}}\omega^2 V_1^{(1)} = -b_1 \frac{dV_1^{(0)}}{dz} - i\alpha b_1 \xi V_1^{(0)} + \frac{\rho_0}{c_{44}} z\omega^2 b_1 V_1^{(0)} - \frac{\rho_0}{c_{44}} z\omega^2 b_2 V_1^{(0)} \tag{4.37}$$

Substituting equation (4.36) and its first derivative into equation (4.37) we have,

$$\frac{d^2 V_1^{(1)}}{dz^2} + 2i\alpha\xi \frac{dV_1^{(1)}}{dz} - \beta\xi^2 \Big(1 - \frac{c^2}{c_1^2}\Big) V_1^{(1)} = -ib_1\xi\Big(- s_1 A_1 e^{-i\xi s_1 z} + s_2 A_2 e^{i\xi s_2 z}\Big) + \Big[- i\alpha b_1 \xi + \frac{\rho_0}{c_{44}}\omega^2 b_1 z - \frac{\rho_0}{c_{44}}\omega^2 b_2 z\Big]\Big[A_1 e^{-i\xi s_1 z} + A_2 e^{i\xi s_2 z}\Big] \tag{4.38}$$

We therefore employ the Green's function approach to solve the resulting in-homogeneous second order ordinary differential equation (4.38)

Let $G_1(z, z_0)$ the Green's function associated with the problem. That is Green's function for z in the layer. $G_1(z, z_0)$ satisfies,

- $LG_1(z, z_0) = 0$, $z \neq z_0$ with homogeneous boundary conditions as $G_1'(-H, z_0) = 0$ and $G_1'(0, z_0) = 0$.

- $G_1(z, z_0)$ is a continuous function of z $(z = z_0)$ that is $G_1(z, z_0)|_{z=z_0^+} = G_1(z, z_0)|_{z=z_0^-}$;

- Jump in derivative, that is, $G'(z, z_0)|_{z=z_0} = \frac{dG}{dz}|_{z=z_0}$ has a jump discontinuity$\Big[G'(z, z_0^+) - G'(z, z_0^-) = -1\Big]$

We introduce Green's function $G_1(z, z_0)$ as the solution of the problem $LV_1 = \delta(z - z_0)$ with homogeneous boundary conditions. $V_1'(0) = 0$ and $V_1'(-H) = 0$ As $G_1(z, z_0)$ is a solution of the above $LG_1(z, z_0) = \delta(z - z_0)$. So $V_1(z_0) = \int_{-H}^{0} G_1(z, z_0)V_1(z)dz$. Using symmetry of operator L we have that $V_1(z) = \int_{-H}^{0} G_1(z, z_0)V_1(z_0)dz_0$. Equation (4.38) is an inhomogeneous problem so we find the general solution as

$$\frac{d^2G_1^{(1)}}{dz^2} + 2i\alpha\xi\frac{dG_1^{(1)}}{dz} - \beta\xi^2\Big(1 - \frac{c^2}{c_1^2}\Big)G_1^{(1)} = \delta(z - z_0). \tag{4.39}$$

Since we have an inhomogeneous problem we first solve,

$$\frac{d^2G_1^{(1)}}{dz^2} + 2i\alpha\xi\frac{dG_1^{(1)}}{dz} - \beta\xi^2\Big(1 - \frac{c^2}{c_1^2}\Big)G_1^{(1)} = 0 \tag{4.40}$$

With linearly independent solutions $e^{-i\xi s_1 z}$ and $e^{i\xi s_2 z}$ where $s_1 = \alpha + \sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)}$ and $s_2 = -\alpha + \sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)}$ so that the Green's function is of the form

$$G_1(z, z_0) = \begin{cases} T_1e^{-i\xi s_1 z} + T_2e^{-i\xi s_2 z}, & -H < z < z_0 \\ \\ Y_1e^{-i\xi s_1 z} + Y_2e^{i\xi s_2 z}, & z_0 < z < 0, \end{cases}$$

Now using the conditions above, $G_1'(-H, z_0) = -s_1T_1e^{-i\xi s_1 z} + s_2T_2e^{i\xi s_2 z} = 0$, So that $T_2 = \frac{s_1}{s_2}T_1e^{i\xi H(s_1+s_2)}$ also $G_1'(0, z_0) = -s_1Y_1 + s_2Y_2 = 0$ and $Y_2 = \frac{s_1}{s_2}Y_1$. Substituting $T_2 = \frac{s_1}{s_2}T_1e^{i\xi H(s_1+s_2)}$ and $Y_2 = \frac{s_1}{s_2}Y_1$ into $G_1(z, z_0)$ we have,

$$G_1(z,z_0)=\begin{cases} T_1\Big(e^{-i\xi s_1 z}+\dfrac{s_1}{s_2}T_1 e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z}\Big), & -H<z<z_0\\ Y_1\Big(e^{-i\xi s_1 z}+\dfrac{s_1}{s_2}e^{i\xi s_2 z}\Big), & z_0<z<0, \end{cases}$$

For the condition that $G_1(z,z_0)|_{z=z_0^+}=G_1(z,z_0)|_{z=z_0^-}$ gives

$$T_1\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}T_1 e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z_0}\Big)=Y_1\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}e^{i\xi s_2 z_0}\Big)$$

$$Y_1=T_1\left[\frac{\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}T_1 e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z_0}\Big)}{\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}e^{i\xi s_2 z_0}\Big)}\right] \tag{4.41}$$

Also from condition $G'(z,z_0^+)-G'(z,z_0^-)=-1$,

$$T_1\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}T_1 e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z_0}\Big)-Y_1\Big(e^{-i\xi s_1 z_0}+e^{i\xi s_2 z_0}\Big)=-\frac{1}{i\xi s_1} \tag{4.42}$$

Put equation (4.41) into equation (4.42) we have

$$Y_1=-\left[\frac{e^{i\xi z_0(s_1+s_2)}\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z_0}\Big)}{i\xi s_1(1+\frac{s_1}{s_2})\Big(e^{i\xi H(s_1+s_2)}-1\Big)}\right] \tag{4.43}$$

$$T_1=-\left[\frac{e^{i\xi z_0(s_1+s_2)}\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}e^{i\xi s_2 z_0}\Big)}{i\xi s_1(1+\frac{s_1}{s_2})\Big(e^{i\xi H(s_1+s_2)}-1\Big)}\right] \tag{4.44}$$

so

$$G_1(z,z_0)=\begin{cases} -\dfrac{e^{i\xi z_0(s_1+s_2)}\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}e^{i\xi s_2 z_0}\Big)}{i\xi s_1(1+\frac{s_1}{s_2})\Big(e^{i\xi H(s_1+s_2)}-1\Big)}\Big(e^{-i\xi s_1 z}+\dfrac{s_1}{s_2}T_1 e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z}\Big), -H<z<z_0 \\ -\dfrac{e^{i\xi z_0(s_1+s_2)}\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z_0}\Big)}{i\xi s_1(1+\frac{s_1}{s_2})\Big(e^{i\xi H(s_1+s_2)}-1\Big)}\Big(e^{-i\xi s_1 z}+\dfrac{s_1}{s_2}e^{i\xi s_2 z}\Big),\ z_0<z<0, \end{cases}$$

$$V_1^{(1)}=\int_{-H}^{0}G_1(z,z_0)fz_0dz_0 \tag{4.45}$$

We need to find the Fourier inverse of $V_1^{(1)}$ as

$$v_1^{(1)}(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-H}^{0}G_1(z,z_0)f(z_0)e^{-i\xi x}dz_0d\xi. \tag{4.46}$$

So that

$$v_1^{(1)}(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-H}^{z}G_1^a(z,z_0)f(z_0)e^{-i\xi x}dz_0d\xi+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{z}^{0}G_1^b(z,z_0)f(z_0)e^{-i\xi x}dz_0d\xi \tag{4.47}$$

Where

$$G_1^a(z,z_0)=-\left[\frac{e^{i\xi z_0(s_1+s_2)}\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}e^{i\xi s_2 z_0}\Big)}{i\xi s_1(1+\frac{s_1}{s_2})\Big(e^{i\xi H(s_1+s_2)}-1\Big)}\right]\Big(e^{-i\xi s_1 z}+\frac{s_1}{s_2}T_1 e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z}\Big),$$

$$G_1^b(z,z_0)=-\left[\frac{e^{i\xi z_0(s_1+s_2)}\Big(e^{-i\xi s_1 z_0}+\frac{s_1}{s_2}e^{i\xi H(s_1+s_2)}e^{i\xi s_2 z_0}\Big)}{i\xi s_1(1+\frac{s_1}{s_2})\Big(e^{i\xi H(s_1+s_2)}-1\Big)}\right]\Big(e^{-i\xi s_1 z}+\frac{s_1}{s_2}e^{i\xi s_2 z}\Big)$$

and

$$f(z_0)=\Big[-ib_1\xi s_1-i\alpha b_1\xi+\frac{\rho_0}{c_{44}}\omega^2 b_1 z_0-\frac{\rho_0}{c_{44}}\omega^2 b_2 z_0\Big]A_1e^{-i\xi s_1 z}+\Big[-ib_1\xi s_2- i\alpha b_1\xi+\frac{\rho_0}{c_{44}}\omega^2 b_1 z_0-\frac{\rho_0}{c_{44}}\omega^2 b_2 z_0\Big]A_2e^{i\xi s_2 z}$$

Therefore

$$v_1 = v_1^{(0)} + \varepsilon v_1^{(1)} + ...$$

gives

$$v_1(x,z) = -\frac{A_1}{2\pi i}\left[\frac{1}{(s_1 z + x)}\right] + \frac{A_2}{2\pi i}\left[\frac{1}{(s_2 z - x)}\right] + \frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\int_{-H}^{0} G_1(z,z_0) f(z_0) e^{-i\xi x} dz_0 d\xi \tag{4.48}$$

4.7 Alternative solution in the half-space

We next find the solution of the anisotropic elastic half space of the above problem in Section 4.1 by using the perturbation series method. We follow similar process as we have in Section 4.3

$$a_{44}\frac{\partial^2 v_2}{\partial z^2} + 2a_{46}\frac{\partial^2 v_2}{\partial x \partial z} + a_{66}\frac{\partial^2 v_2}{\partial x^2} = \rho_2 \frac{\partial^2 v_2}{\partial t^2}$$

Now assuming the time harmonic variation is taken as $e^{i\omega t}$ and be suppressed throughout. That is $v_2(x,z,t) = v_2(x,z)e^{i\omega t}$.

$$a_{44}\frac{\partial^2 v_2}{\partial z^2} + 2a_{46}\frac{\partial^2 v_2}{\partial x \partial z} + a_{66}\frac{\partial^2 v_2}{\partial x^2} + \rho_2 \omega^2 v_2 = 0$$

Dividing throughout by a_{44} we have

$$\frac{\partial^2 v_2}{\partial z^2} + 2q\frac{\partial^2 v_2}{\partial x \partial z} + b\frac{\partial^2 v_2}{\partial x^2} + \frac{\rho_2}{a_{44}}\omega^2 v_2 = 0 \tag{4.49}$$

Where $q = \dfrac{a_{46}}{a_{44}}$ and $b = \dfrac{a_{66}}{a_{44}}$. Now employing the perturbation series we let

$$v_2 = v_2^{(0)} + \varepsilon v_2^{(1)} + \varepsilon^2 v_2^{(2)} + ... \tag{4.50}$$

Substituting equation (4.50) into equation (4.49) we have,

$$\frac{\partial^2}{\partial z^2}[v_2^{(0)} + \varepsilon v_2^{(1)} + \varepsilon^2 v_2^{(2)} + ...] + 2q\frac{\partial^2}{\partial x \partial z}[v_2^{(0)} + \varepsilon v_2^{(1)} + \varepsilon^2 v_2^{(2)} + ...] + b\frac{\partial^2 v_2}{\partial x^2} + \frac{\rho_2}{a_{44}}\omega^2[v_2^{(0)} + \varepsilon v_2^{(1)} + \varepsilon^2 v_2^{(2)} + ...] = 0$$

Comparing coefficients of ε we have

$$\frac{\partial^2 v_2^{(0)}}{\partial z^2} + 2q\frac{\partial^2 v_2^{(0)}}{\partial x \partial z} + b\frac{\partial^2 v_2^{(0)}}{\partial x^2} + \frac{\rho_2}{a_{44}}\omega^2 v_2^{(0)} = 0 \tag{4.51}$$

$$\frac{\partial^2 v_2^{(1)}}{\partial z^2} + 2q\frac{\partial^2 v_2^{(1)}}{\partial x \partial z} + b\frac{\partial^2 v_2^{(1)}}{\partial x^2} + \frac{\rho_2}{a_{44}}\omega^2 v_2^{(1)} = 0 \tag{4.52}$$

Now solving equations (4.51) and equation (4.52) we first take the Fourier transform pair with respect to x as

$$\begin{aligned} V_2^{(j)}(\xi, z) &= \int_{-\infty}^{\infty} v_1^{(j)}(x, z)e^{i\xi x}dx \\ v_2^{(j)}(x, z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} V_1^{(j)}(\xi, z)e^{-i\xi x}d\xi \\ j &= 0, 1 \end{aligned} \tag{4.53}$$

$$\frac{d^2 V_2^{(0)}}{dz^2} + 2qi\xi\frac{d^2 V_2^{(0)}}{dz} - b\xi^2\left(1 - \frac{c^2}{c_2^2}\right)V_2^{(0)} = 0 \tag{4.54}$$

Where $c_2^2 = \frac{\xi^2 a_{66}}{k^2 \rho_2}$ and $\omega = kc$. Solving for $V_2^{(0)}$ in the second order ordinary differential equation (77) we have that,

$$V_2^{(0)} = C_1 e^{-i\xi h_1 z} + C_2 e^{i\xi h_2 z} \tag{4.55}$$

Where $h_1 = q + \sqrt{q^2 + b\Big(\frac{c^2}{c_2^2} - 1\Big)}$ and $h_2 = -q + \sqrt{q^2 + b\Big(\frac{c^2}{c_2^2} - 1\Big)}$. Since we are in the lower half space the displacement with $C_2 = 0$ is,

$$V_2^{(0)} = C_1 e^{-i\xi h_1 z} \tag{4.56}$$

Now the Fourier inverse is given as

$$v_2^{(0)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_1 e^{-i\xi h_1 z} e^{-i\xi x} d\xi = \frac{C_1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi(h_1 z + x)} d\xi$$

We assume x is large so that $(h_1 z + x) > 0$ we have

$$v_2^{(0)} = -\frac{C_1}{2\pi i(h_1 z + x)} \tag{4.57}$$

Similarly, $v_2^{(1)} = -\dfrac{D_1}{2\pi i(h_1 z + x)}$, $v_2^{(2)} = -\dfrac{E_1}{2\pi i(h_1 z + x)}$ and so on.

Hence

$$v_2 = -\frac{C_1}{2\pi i(h_1 z + x)} - \frac{\varepsilon D_1}{2\pi i(h_1 z + x)} - \frac{\varepsilon^2 E_1}{2\pi i(h_1 z + x)} + ... \tag{4.58}$$

4.8 Boundary conditions for $\varepsilon = 0$

The appropriate boundary conditions for the propagation of Love wave are given as follows with the assumption that the inhomogeneous anisotropic elastic layer and the anisotropic elatic half space are perfectly bonded.

1. At the free surface $z = -H$, $(\sigma_{yz})_{medium1} = 0$ as the normal component of stress vanishes. $(\sigma_{yz})_{medium\ 1} = c_{44}\dfrac{\partial v_1^{(0)}}{\partial z} + c_{46}\dfrac{\partial v_1^{(0)}}{\partial x} = 0$.

2. At the interface $z = 0$, the displacement components are continuous that is $v_1 = v_2$ that is $v_1^{(0)} = v_2^{(0)}$.

3. At the interface $z = 0$, the stress is continuous, that is, $(\sigma_{yz})_{medium\ 1} = (\tau_{yz})_{medium\ 2}$, That is, $c_{44}\dfrac{\partial v_2^{(0)}}{\partial z} + c_{46}\dfrac{\partial v_1^{(0)}}{\partial x} = a_{44}\dfrac{\partial v_2^{(0)}}{\partial z} + a_{46}\dfrac{\partial v_2^{(0)}}{\partial x}$.

4.9 Dispersion relation for $\varepsilon = 0$

We establish a dispersion relation for the case when $\varepsilon = 0$ using boundary conditions in Section 4.8 as follows,

$$A_1 + A_2 - C_1 = 0 \tag{4.59}$$

$$[c_{44}s_1 + c_{46}]A_1 + [c_{46}s_1 - c_{44}s_2]A_2 - (a_{44}h_1 + a_{46})C_1 = 0 \tag{4.60}$$

$$(-s_2H - x)^2(c_{44}s_1 + c_{46})A_1 + (x - s_1H)^2(c_{46}s_1 - c_{44}s_2)A_2 = 0 \tag{4.61}$$

The dispersion relation can be obtained in a determinant form by eliminating the constants A_1, A_2 and C_1 which gives

$$\begin{vmatrix} 1 & 1 & -1 \\ (c_{44}s_1 + c_{46}) & (c_{46}s_1 - c_{44}s_2) & -(a_{44}h_1 + a_{46}) \\ (x + s_2H)^2(c_{44}s_1 + c_{46}) & (x - s_1H)^2(c_{46}s_1 - c_{44}s_2) & 0 \end{vmatrix} = 0 \quad (4.62)$$

Equation (4.62) is the dispersion relation for Love waves in the special case when $\varepsilon = 0$. The non trivial solution of these equations $|d_{mn}| = 0$, $\forall\ m,\ n = 1, 2, 3$. Where d_{mn} are the entries of determinant given above.

CHAPTER 5

NONLINEAR LOVE WAVES IN ISOTROPIC MATERIALS

In this chapter we study nonlinear Love waves in an isotropic material. Various types of nonlinear models have been studied. For large deformation, the Murnaghan model is widely used due to the fact that the potential associated with this Murnaghan model has a third algebraic invariant that makes it possible to take into consideration numerous essential wave effects. The method of successive approximation is used in the analysis of the cubic nonlinear Love wave equation. Materials that are deformed elastically are basically classified into hypoelastic, generally elastic and hyperelastic materials. For large deformations we observe that the Cauchy-Green strain tensor is in nonlinear relationship to the displacement vector.The Cauchy-Green strain tensor is given by a displacement vector in the reference configuration and uses the Lagrangian coordinate system:

$$\varepsilon_{qr} = \frac{1}{2}(u_{q,r} + u_{r,q} + u_{i,q}u_{i,r}) \tag{5.1}$$

The Almansi strain tensor is given by a known displacement vector in the actual configuration and uses the Eulerian coordinate system. In nonlinear mechanics of materials, the representation by means of invariants are often used. It is very relevant we realize that in the process of changing from the linear elastic model to nonlinear models (when describing anisotropic materials), the material mechanics face challenges. Some nonlinear models in the isotropic hyperelastic materials are Seth model, John model, Signorini and Murnaghan models.

5.1 Seth model

This is the simplest (two constant) nonlinear model. The stress-strain relationship in this model corresponds to that of the classical form of the generalized Hooke's law, in which the two Lame elastic constants are kept and the infinitesimally small strains are altered in the finite strains. The Seth's Model is given as

$$\tau_{qr} = \lambda \tilde{\epsilon_{rr}} \delta_{qr} + 2\mu \tilde{\epsilon_{qr}} \tag{5.2}$$

Where τ_{qr} is a Kirchhoff stress tensor which is the nonlinear term on the left hand side and on the right hand side, the nonlinear Almansi strain tensor $\tilde{\epsilon_{qr}}$. Seth model does not have potential (internal energy) that means it lacks the core property of hyperelasticity.

5.2 John model

The John model is also two constant model that has a potential which denotes the geometrically nonlinear case and abandons the physical nonlinearity case. The potential is normally expressed as

$$\theta = \tfrac{1}{2}\lambda(r_1)^2 + \mu(r_2)$$

Where λ and μ are the Lame's constants; (r_1) and (r_2) are the first and second basis invariants of the Cauchy-Greens strain tensor.

5.3 Signorini model

Since the Seth's model does not have the core property of hyperelasticity, that is the potential associated with Seth model cannot be written, Signorini introduced a model to correct this deficiency in the Seth's model. The Signorini model provides a connection between the stress tensor and the Almansi strain tensor as well. It is a three constant model and its potential is quadratically nonlinear. This model has a potential called the Signorini potential which is given in the natural reference configuration. The Signorini potential is written as

$$W(\epsilon_{\tilde{nm}}) = \sqrt{\frac{G}{g}}\left\{PJ_2(\tilde{\epsilon})+\frac{1}{2}\left(\lambda+\mu-\frac{P}{2}\right)+\left(\mu+\frac{P}{2}\right)\left(J-J_1(\tilde{\epsilon})\right)-\left(\mu+\frac{P}{2}\right)\right\} \tag{5.3}$$

Where $J_k(\tilde{\epsilon})$ represents invariants of the Almansi strain tensor. We note that λ and μ are the Lame constants and P represents one of the three Murnaghan constants. The next potential which is widely used in nonlinear wave models is the Murnaghan potential.

5.4 Murnaghan model

We now consider the next type of potential which is a cubic potential called the Murnaghan potential proposed by Murnaghan for the Cauchy-Green strain tensor given by

$$W(\varepsilon_{qr}) = \frac{1}{2}\lambda(\varepsilon_{mm})^2 + \mu(\varepsilon_{qr})^2 + \frac{1}{3}P\varepsilon_{qr}\varepsilon_{qm}\varepsilon_{rm} + Q(\varepsilon_{qr})^2\varepsilon_{mm} + \frac{1}{3}R(\varepsilon_{mm})^3 \quad (5.4)$$

Where λ, μ are the 2^{nd} order Lame Constants ; P, Q and R the Murnaghan 3^{rd} order elastic constants. This is the most widely used nonlinear model because the Murnaghan potential has a third algebraic invariant that makes it possible to take into consideration numerous essential wave effects. We can write the Murnaghan potential through the first algebraic invariants (I_k) of the strain tensor ε_{qr} as;

$$W(I_1, I_2, I_3) = \frac{1}{2}\lambda I_1^2 + \mu I_2 + \frac{1}{3}PI_3 + QI_1I_2 + \frac{1}{3}RI_1^3 \quad (5.5)$$

The Murnaghan Potential describes the hyperelastic deformation. After substituting (5.1) into (5.4), we have quite a number of submodels of the Murnaghan model [50]. We notice also after the substitution that, the expressions which involved second order and third order strain tensor parts has transformed into expressions including the second to sixth order components of the displacement $u_{q,r}$ gradient.

We would consider the summands up to the third powers neglecting the fourth to the sixth powers for now. This leads to a general form of the resulting Murnaghan potential as

$$W = \frac{1}{2}\lambda(u_{m,m})^2 + \frac{1}{4}\mu(u_{q,r} + u_{r,q})^2 + (\mu + \frac{1}{4}P)u_{q,r}u_{m,q}u_{m,r} + \frac{1}{2}(\lambda + Q)u_{m,m}(u_{q,r})^2 \\ + \frac{1}{12}Pu_{q,r}u_{r,m}u_{m,m} + \frac{1}{2}Qu_{q,r}u_{r,q}u_{m,m} + \frac{1}{3}R(u_{m,m})^3 \tag{5.6}$$

Many subpotentials and subsubpotentials are derived from the (5.6). For our work we would consider the subsubpotential corresponding to Love wave. For this potential corresponding to Love wave, the resulting problem is normally declared for elastic media whose mechanical state depend on two spatial coordinates (x, z) only which is characterized by only one component of the displacement vector u_3.

5.5 Method of Successive Approximation

Two fundamental methods are used in solving all elastic wave equations. These are the slowly varying amplitudes approach and the successive approximation approach. For these two methods to be efficient, an assumption is made on the weak nonlinearity of the elastic medium, that is wave features such as wavelengths and amplitudes and elastic constants have some limitations. The successive approximations approach also called the perturbation method or the method of small parameter was used by Earnshow to solve a simple wave equation with finite amplitude obtained from a radiation problem. The perturbation method is useful when the solution of the problem (closed problem) is known. As an illustration we outline the steps involved in using the perturbation method by considering a non-linear harmonic plane longitudinal wave equation obtained from the Murnaghan model given by;

$$u_{1,tt} - (v_L)^2 u_{1,11} = (\frac{N_1}{\rho}) u_{1,11} u_{1,1} \tag{5.7}$$

Where $v_L = \sqrt{\dfrac{\lambda + 2\mu}{\rho}} = \dfrac{\omega}{k_L}$.

The linear equation

$$u_{1,tt} - (v_L)^2 u_{1,11} = 0 \tag{5.8}$$

is a closed problem with a known classical solution. We introduce a perturbation term ε which is usually small into the nonlinear wave equation in such a way that when $\varepsilon = 1$, the equation must be the same as (5.7) and when $\varepsilon = 0$, it must match with (5.8). Hence we must have,

$$u_{1,tt} - (v_L)^2 u_{1,11} = \varepsilon(\frac{N_1}{\rho}) u_{1,11} u_{1,1}. \tag{5.9}$$

Now according to the perturbation method, the solution of (5.9); $u_1(x, t, \varepsilon)$ can be found to be a convergent series

$$u_1(x, t, \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r u_1^r(x, t) \tag{5.10}$$

We note that $u_1^{(0)}(x,t)$ is a solution to (5.8); a linear equation. Substituting (5.10) into (5.9) gives a recurrent equation for any approximation ;

$$u_{1,tt}^{(r)} - (v_L)^2 u_{1,11}^{(r)} = \Big(\frac{N_1}{\rho}\Big) u_{1,11}^{(r-1)} u_{1,1}^{(r-1)} \tag{5.11}$$

Hence the solution of (5.7) is denoted by the infinite sum of approximations given as;

$$u_1(x,t) = u_1(x,t,\varepsilon=1) = u_1^{(1)}(x,t) + u_1^{(2)}(x,t) + u_1^{(3)}(x,t) + ... = \sum_{r=0}^{\infty} u_1^r(x,t) \tag{5.12}$$

The solution to wave equation is obtained by solving first the linear approximation, the second approximation and so on [57], [59], [61], [67]. Nonlinear elastic Love wave derived from the Murnaghan potential model is solved using the successive approach up to the second approximation as $u_3^{l(h)}(x,z,t) = u_3^{(0)}(x,z,t) + u_3^{(1)}(x,z,t)$

5.6 Statement of the nonlinear Love wave problem

We consider the problem on the Love wave in the classical statement but this has added assumptions on the nonlinearity of the deformation process. Geometrically, the nonlinear problem statement agrees in certain parts with the linear case. We consider a layer (medium 1) of constant thickness $-H \leq z \leq 0$ and the upper half-space (medium 2) $z \geq 0$ described in the cartesian coordinates $Oxyz$. The $z - axis$ is taken vertically downward deep into the half-space and the interface. From the mechanics point of view, we include few assumptions as follows

1. Both the half-space (medium 2) and layer(medium 1) comprise of nonlinear elastic materials with contrasting properties. We note that from henceforth we will describe the elastic half-space with index h and index l as the layer.

2. The Murnaghan model is used to describe the deformation of materials. That is, density $\rho_{l(h)}$ and $\lambda_{l(h)}$, $\mu_{l(h)}$, $P_{l(h)}$ $Q_{l(h)}$ and $R_{l(h)}$.

3. Again the half-space and layer are perfectly welded together (welded contact). The stress components and displacements are at the interface $z = 0$ are continuous and the lower layer is free of stresses.

Now we consider the horizontally polarized shear wave for which both longitudinal and vertical displacements u_1 and u_2 respectively are zero. Hence the only possibility of propagation of the wave is along the $Oz - axis$ direction near the interface between the half-space and layer. The wave is thus represented as

$$u_3^{l(h)} = U_3^{l(h)}(z)e^{i(kx-\omega t)} \tag{5.13}$$

Where $U_3^{l(h)}(z)$ is unknown amplitude and k is the wave number. If the wave is restricted or localized near the interface $z = 0$, that is, it has the maximum amplitude at the interface and when there is an increase in the depth z the amplitude decays sharply, then the above problem statement in linear elasticity agrees with the problem of Love wave propagation. To describe the medium in which a wave with such characteristics propagates, we will employ the Murnaghan model of a nonlinear elastic material whose mechanical form relies on two spatial variables (x, z) and its characterized by only one component of the displacement vector u_3. We use the symmetric Cauchy-Green strain tensor ε_{qr}, the nonsymmetric Kirchhoff stress tensor (τ_{qr}) and the displacement gradients $u_{q,r}$ to describe the Murnaghan potential. Even though there exists nine components of the displacement gradients $u_{q,r}$ but only two components $(u_{3,1})$ and $(u_{3,2})$ are non-zero. The strain components can be found using the formula below

$$\varepsilon_{qr} = \frac{1}{2}(u_{q,r} + u_{r,q} + u_{i,q}u_{i,r})$$

the components of the strain can be found from the following formulas

$$\varepsilon_{11} = u_{1,1} + \frac{1}{2}(u_{1,1}u_{1,1} + u_{2,1}u_{2,1} + u_{3,1}u_{3,1}) = \frac{1}{2}(u_{3,1})^2$$

$$\varepsilon_{22} = u_{2,2} + \frac{1}{2}(u_{1,2}u_{1,2} + u_{2,2}u_{2,2} + u_{3,2}u_{3,2}) = \frac{1}{2}(u_{3,2})^2$$

$$\varepsilon_{33} = \frac{1}{2}(u_{3,3} + u_{3,3} + u_{k,3}u_{k,3}) = 0$$

$$\varepsilon_{12} = \tfrac{1}{2}(u_{1,2} + u_{2,1} + u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}) = \tfrac{1}{2}u_{3,1}u_{3,2}$$

$$\varepsilon_{13} = \frac{1}{2}(u_{1,3} + u_{3,1} + u_{1,1}u_{1,3} + u_{2,1}u_{2,3} + u_{3,1}u_{3,3}) = \frac{1}{2}u_{3,1}$$

$$\varepsilon_{23} = \frac{1}{2}(u_{2,3} + u_{3,2} + u_{1,2}u_{1,3} + u_{2,2}u_{2,3} + u_{3,2}u_{3,3}) = \frac{1}{2}u_{3,2}$$

The Murnaghan subsubmodel corresponding to the Love wave has a potential given as

$$W = \frac{1}{4}\lambda\Big[(u_{3,1})^2 + (u_{3,2})^2\Big]^2 + \mu\Big[\frac{1}{2}(u_{3,1})^2 + \frac{1}{2}(u_{3,2})^2 + \frac{1}{4}(u_{3,1})^4 + \frac{1}{4}(u_{3,2})^4 +$$
$$\frac{1}{4}(u_{3,1}u_{3,2})^2\Big] + \frac{1}{24}P\Bigg\{3\Big[(u_{3,1})^2 + (u_{3,2})^2\Big]^2 + (u_{3,1})^6 + (u_{3,2})^6$$
$$+3(u_{3,1})^2(u_{3,2})^2\Big[(u_{3,1})^2 + (u_{3,2})^2\Big]\Bigg\} + \frac{1}{8}Q\Bigg[2(u_{3,1})^2 + 2(u_{3,2})^2$$
$$+(u_{3,1})^4 + (u_{3,2})^4 + (u_{3,1}u_{3,2})^2\Bigg]\Big[(u_{3,1})^2 + (u_{3,2})^2\Big] +$$
$$\frac{1}{24}R\Big[(u_{3,1})^2 + (u_{3,2})^2\Big]^3 \tag{5.14}$$

Here we notice from equation (5.14) above consists of only even powers of the two components of displacement gradient $u_{3,1}u_{3,2}$. It also has only second powers corresponding to linear elasticity, the fourth powers corresponding to the cubically nonlinear elasticity theory and the sixthpowers corresponding to the nonlinearity of the fifth order.

Next, We consider the asymmetric Kirchhoff stress tensor τ_{qr} which does not include all the components is usually used to describe the Murnaghan potential. The classical asymmetric Kirchhoff stress tensor for hyperelastic material is defined as

$$\tau_{qr} = \frac{\partial W}{\partial u_{r,q}} \tag{5.15}$$

We again realize that since the expression of the potential include only two of the nine components of the deformation gradients, the stress tensor also has only two components which are not zero.

From the equation of motion formula given as

$$\sigma_{qr,q} + X_r = \rho \ddot{u}_r \tag{5.16}$$

where σ_{qr} is the symmetric Cauchy stress tensor and X_r is the body force. Replacing the symmetric Cauchy-Green stress tensor σ_{qr} by an asymmetric Kirchhoff tensor τ_{qr} we have,

$$\tau_{qr,q} + X_r = \rho \ddot{u}_r \tag{5.17}$$

For our problem we only consider the constitutive equations τ_{13} and τ_{23} as below;

$$\begin{aligned}\tau_{13} = \mu u_{3,1} + (\lambda+\mu)\Big[(u_{3,1})^3 + \frac{1}{2}u_{3,1}(u_{3,2})^2\Big] + \frac{1}{4}P\Big\{2u_{3,1}\Big[(u_{3,1})^2 + (u_{3,2})^2\Big] + \\ (u_{3,1})^5 + u_{3,1}(u_{3,2})^2\Big[2(u_{3,1})^2 + (u_{3,2})^2\Big]\Big\} + \frac{1}{4}Q\Big\{\Big[2(u_{3,1})^3 + 2u_{3,1}(u_{3,2})^2 + \\ (u_{3,1})^5 + u_{3,1}(u_{3,2})^4 + (u_{3,1})^3(u_{3,2})^2\Big]\Big\} + \frac{1}{4}Ru_{3,1}\Big[(u_{3,1})^2 + 2(u_{3,2})^2\Big]^2\end{aligned} \tag{5.18}$$

$$\begin{aligned}\tau_{23} = \mu u_{3,2} + (\lambda+\mu)\Big[(u_{3,2})^3 + \frac{1}{2}u_{3,2}(u_{3,1})^2\Big] + \frac{1}{4}P\Big\{2u_{3,2}\Big[(u_{3,1})^2 + (u_{3,2})^2\Big] + \\ (u_{3,2})^5 + u_{3,2}(u_{3,1})^2\Big[2(u_{3,2})^2 + (u_{3,1})^2\Big]\Big\} + \frac{1}{4}Q\Big\{\Big[2(u_{3,2})^3 + 2u_{3,2}(u_{3,1})^2 + \\ (u_{3,2})^5 + u_{3,2}(u_{3,1})^4 + (u_{3,2})^3(u_{3,1})^2\Big]\Big\} + \frac{1}{4}Ru_{3,2}\Big[(u_{3,1})^2 + 2(u_{3,2})^2\Big]^2\end{aligned} \tag{5.19}$$

5.7 Nonlinear Love wave equation.

We formulate the nonlinear equation from the equation of motion as described earlier as

$$\tau_{qr,q} + X_r = \rho \ddot{u}_r \quad \text{for } r, q = 1, 2, 3$$

assuming there are no body forces

$$\tau_{qr,q} = \rho \ddot{u}_r \quad \text{for } r, q = 1, 2, 3.$$

for $r = 1$ and $q = 1, 2, 3$

$$\tau_{q1,q} = \rho \ddot{u}_1,$$

$$\tau_{11,1} + \tau_{21,2} + \tau_{31,3} = \rho \ddot{u}_1. \tag{5.20}$$

for $r = 2$ and $q = 1, 2, 3$

$$\tau_{q2,q} = \rho \ddot{u}_2$$

$$\tau_{12,1} + \tau_{22,2} + \tau_{32,3} = \rho \ddot{u}_2 \tag{5.21}$$

for $r = 3$ and $q = 1, 2, 3$

$$\tau_{q3,q} = \rho \ddot{u}_3,$$

$$\tau_{13,1} + \tau_{23,2} + \tau_{33,3} = \rho \ddot{u}_3. \tag{5.22}$$

Two of the three equations of motion (5.20) and (5.21) are identically zero and the only nonzero equation is given as;

$$\tau_{13,1} + \tau_{23,2} = \rho \ddot{u}_3 \tag{5.23}$$

Substituting τ_{13} and τ_{23} in the equation of motion in equation (5.23) and simplifying gives

$$\rho\ddot{u}_3 - \mu(u_{3,11} + u_{3,22}) =$$

$$S_1(u_{3,1})^2u_{3,11} + S_2(u_{3,2})^2u_{3,11} + S_1(u_{3,2})^2u_{3,22} + S_2(u_{3,1})^2u_{3,22} + 4S_2u_{3,1}u_{3,2}u_{3,12}$$

$$+T_1(u_{3,1})^4u_{3,11} + T_1(u_{3,2})^4u_{3,22} + T_2(u_{3,2})^4u_{3,11} + T_2(u_{3,1})^4u_{3,22}$$

$$+T_3(u_{3,1})^3u_{3,2}u_{3,12} + T_3u_{3,1}(u_{3,2})^3u_{3,12} + T_4(u_{3,1})^2(u_{3,2})^2u_{3,11} + T_4(u_{3,2})^2(u_{3,1})^2u_{3,22} \tag{5.24}$$

Where

$$S_1 = 3\Big[(\lambda+\mu)+\tfrac{1}{4}P+\tfrac{1}{2}Q\Big], \quad S_2 = \tfrac{1}{2}\Big[(\lambda+\mu)+P+Q\Big], \quad T_1 = \tfrac{5}{4}\Big[P+Q+R\Big],$$

$$T_2 = \Bigg[P + \tfrac{1}{4}Q + \tfrac{1}{4}R\Bigg], \quad T_3 = \Big[2P + \tfrac{3}{2}Q + 2R\Big], \quad T_4 = \tfrac{3}{4}\Big[2P + Q + 2R\Big].$$

We could see that equation (5.24) contains only nonlinear terms of the third order with five terms and eight terms fifth order. This is as a result of the statement of the problem which includes the presence of nonlinear terms associated with the presence of nonlinearity of the deformation process which is also allowed in the description of physical nonlinearity in kinematic equations. We retain only the cubic nonlinearity as in equation (5.24) and we obtain

$$\rho\ddot{u}_3 - \mu(u_{3,11} + u_{3,22}) =$$

$$S_1(u_{3,1})^2u_{3,11} + S_2(u_{3,2})^2u_{3,11} + S_1(u_{3,2})^2u_{3,22} + S_2(u_{3,1})^2u_{3,22} + 4S_2u_{3,1}u_{3,2}u_{3,12} \tag{5.25}$$

We solve the nonlinear Love wave equation in equation (5.25) using the perturbation method (method of successive approximation) up to the second approximation.

5.7.1 First Approximate solution to the nonlinear Love wave equation.

We notice that the first approximation coincides with the solution of the linear equation as described in Section (1.5). That is the solution in both the Layer and half-space as are given as.

$$u_3^{h(0)}(x,z,t) = l_h e^{-\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}kz} e^{i(kx-\omega t)}, \quad x \in (-\infty,\infty), z \in [0,\infty) \tag{5.26}$$

$$\begin{aligned} u_3^{l(0)}(x,z,t) = l_h \Bigg\{ & -\frac{\mu_h\sqrt{\left[1-\left(v/v_T^h\right)^2\right]}}{\mu_l\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}} sin\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}kz \\ & +cos\sqrt{\left[\left(v/v_T^l\right)^2-1\right]}kz\Bigg\} e^{i(kx-\omega t)},\ x \in (-\infty,\infty),\ z \in [-H,0). \end{aligned} \tag{5.27}$$

5.7.2 Second approximate solution to nonlinear Love wave equation.

To search for the second approximation $u_3^1(x,z,t)$, we need to find the solution of an inhomogeneous linear wave equation with the terms on the right hand side known. We note that the right hand side is known by substitution of the first approximation $u_3^0(x,z,t)$.

Let $\hat{S}_r = \frac{S_1}{\mu}$ for $r = 1,2$, $\frac{\mu}{\rho} = (\frac{1}{v^T})^2$ and $u_3^0(x,z,t)$ be the first approximation, then from equation (5.25) we have

$$(1/v^T)^2\ddot{u}_3^{(1)} - (u_{3,11}^{(1)} + u_{3,22}^{(1)}) =$$

$$\hat{S}_1(u_{3,1}^{(0)})^2u_{3,11}^{(0)} + \hat{S}_2(u_{3,2}^{(0)})^2u_{3,11}^{(0)} + \hat{S}_1(u_{3,2}^{(0)})^2u_{3,22}^{(0)} + \hat{S}_2(u_{3,1}^{(0)})^2u_{3,22}^{(0)} + 4\hat{S}_2u_{3,1}^{(0)}u_{3,2}^{(0)}u_{3,12}^{(0)} \tag{5.28}$$

Equation (5.28) is decomposed into half-space equation and layer equation due to the fact that the solutions for layer and half-space as we have in (5.26) and (5.27) differ. It is therefore pertinent to first evaluate the right hand side of equation (5.28). Doing so and simplifying we have

$$(1/v_T^h)^2\ddot{u}_3^{h(1)} - (u_{3,11}^{h(1)} + u_{3,22}^{h(1)}) = (l_h)^3k^4A_h^{(1)}e^{-3\beta_h kz}e^{3i(kx-\omega t)} \tag{5.29}$$

Where $A_h^{(1)} = -6\hat{S}_2^h(\beta_h)^2 + \hat{S}_1^h[(\beta_h)^4 + 1]$ and $\hat{S}_r^h = \frac{S^{l(h)}}{\mu_{l(h)}}$

Also

$$(1/v_T^l)^2\ddot{u}_3^{l(1)} - (u_{3,11}^{l(1)} + u_{3,22}^{l(1)}) =$$

$$\begin{aligned}\frac{1}{4}(l_h)^3k^4e^{3i(kx-\omega t)}\Big[&W_{l1d}^{(1)}cos\sqrt{[(v/v_T^l)^2-1]}kz + W_{l1c}^{(1)}sin\sqrt{[(v/v_T^l)^2-1]}kz + \\ &W_{l3d}^{(1)}cos3\sqrt{[(v/v_T^l)^2-1]}kz + W_{l3c}^{(1)}sin3\sqrt{[(v/v_T^l)^2-1]}kz\Big]\end{aligned} \tag{5.30}$$

Where

$$W_{l1d}^{(1)} = \Big\{\hat{S}_1^l[(\beta_l)^4(\Phi^2+1) + 3\Phi^2 - 3] + 3\hat{S}^l{}_2(\beta_l)^2[8\Phi^2 - 4]\Big\}$$

$$W_{l1c}^{(1)} = \Phi\Big\{-\hat{S}_1^l[\Phi^2((\beta_l)^4 - 3) + 3(\beta_l)^4 - 2(\beta_l)^4 - 3] - \hat{S}^l{}_2(\beta_l)^2[8\Phi^2 - 22]\Big\}$$

$$W_{l3c}^{(1)} = \Phi\Big\{\hat{S}_1^l[-\Phi^2((\beta_l)^4 + 1) + 3(\beta_l)^4 - 3] + \hat{S}^l{}_2(\beta_l)^2[-4\Phi^2 + 8]\Big\}$$

$$W_{l3d}^{(1)} = \Big\{\hat{S}_1^l[3\Phi^2((\beta_l)^4 + 1) - (\beta_l)^4 - 1] + \hat{S}^l{}_2(\beta_l)^2[-18\Phi^2 + 6]\Big\}$$

Where $\Phi = \frac{\mu_h}{\mu_l}\frac{\beta_h}{\beta_l}$. The right hand side of the inhomogeneous linear equations (5.29) and (5.30) are solutions of the respective homogeneous linear equations which gives a resonant case. There is significant differences between equations (5.29) and (5.30) that is, equation (5.29) contains the third harmonics in z on the right hand side and equation (5.30) has only the first and third harmonics [59]. The solutions of equations (5.29) and (5.30) are given below in the form

$$u_3^{h(1)} = \left[\frac{xz\left[\sqrt{(1-(v/v_T^h)^2)}x + iz\right]}{[1-(v/v_T^h)^2]x^2 + z^2}\right] Y_h^{(1)} e^{-3\beta_h kz} e^{3i(kx-\omega t)} \tag{5.31}$$

Where

$$Y_h^{(1)} = \frac{1}{6}(l_h)^3 k^3 A_h^{(1)} = \frac{1}{6}(l_h)^3 k^3 \Big\{ \hat{S_1^h}[(1-(v/v_T^h)^2)^2 + 1] - 6\hat{S_1^h}(1-(v/v_T^h)^2) \Big\}$$

Also

$$u^{l(1)}(x,z,t) = xz\frac{(l_h)^3 k^3}{24}\Big\{ Y_{1d} cos\sqrt{[(v/v_T^l)^2-1]}kz + Y_{1c} sin\sqrt{[(v/v_T^l)^2-1]}kz \Big\} +$$
$$\Big\{ Y_{1d} cos3\sqrt{[(v/v_T^l)^2-1]}kz + Y_{3c} sin3\sqrt{[(v/v_T^l)^2-1]}kz \Big\} e^{3i(kx-\omega t)} \tag{5.32}$$

Where

$$Y_{1d} = \frac{3}{\Big\{\sqrt{[(v/v_T^l)^2-1]}x^2 - 9(z)^2\Big\}} \Big\{ \sqrt{[(v/v_T^l)^2-1]} W_{l1c}^{(1)} x + 3i W_{l1d}^{(1)} z \Big\}$$

$$Y_{1c} = \frac{3}{\Big\{\sqrt{[(v/v_T^l)^2-1]}x^2 - 9(z)^2\Big\}} \Big\{ \sqrt{[(v/v_T^l)^2-1]} W_{l1c}^{(1)} x - 3i W_{l1d}^{(1)} z \Big\}$$

$$Y_{3c} = \frac{1}{\Big\{\sqrt{[(v/v_T^l)^2-1]}x^2 - (z)^2\Big\}} \Big\{ - W_{l3d}^{(1)} \sqrt{[(v/v_T^l)^2-1]} x - i W_{l3c}^{(1)} z \Big\}$$

$$Y_{3d} = \frac{1}{\Big\{\sqrt{[(v/v_T^l)^2-1]}x^2-(z)^2\Big\}}\Big\{W_{l3c}^{(1)}\sqrt{[(v/v_T^l)^2-1]}x - iW_{l3d}^{(1)}z\Big\}$$

The solution of the above problem statement up to the second approximation is as follows

$$u_3^h(x,z,t) = u_3^{h(0)}(x,z,t) + u_3^{h(1)}(x,z,t) \text{ and } u_3^l(x,z,t) = u_3^{l(0)}(x,z,t) + u_3^{l(1)}(x,z,t)$$

$$\begin{aligned} u_3^h(x,z,t) = l_h e^{-\beta_l kz} e^{i(kx-\omega t)} + \Big[\frac{xz\big[\beta_h x + iz\big]}{(\beta_h)^2x^2+z^2}\Big]Y_h^{(1)}e^{-3\beta_h kz}e^{3i(kx-\omega t)}, \\ x\in(-\infty,\infty), z\in[0,\infty). \end{aligned} \tag{5.33}$$

$$\begin{aligned} u^l(x,z,t) = l_h\Bigg\{-\frac{\mu_h\beta_h}{\mu_l\beta_l}sin\beta_l kz + cos\beta_l kz\Bigg\}e^{i(kx-\omega t)} \\ +xz\frac{(L_h)^3k^3}{24}\Big\{Y_{1d}cos\beta_l kz + Y_{1c}sin\beta_l kz\Big\}+ \\ \Big\{Y_{1d}cos3\beta_l kz + Y_{3c}sin3\beta_l kz\Big\}e^{3i(kx-\omega t)}, \\ x\in(-\infty,\infty), z\in[-H,0]. \end{aligned} \tag{5.34}$$

Where $(\beta_h)^2 = 1-\Big(v/v_T^h\Big)^2$ and $(\beta_l)^2 = \Big(v/v_T^l\Big)^2-1$. It is observed that equations (5.33) and (5.34) have unknown parameters of the linear solution, amplitude l_h and wave number k. This is typical feature of all solutions obtained through the perturbation method. When the amplitude is considered arbitrary, then we could deduce from previous literature that Love wave is a running surface wave or a traveling wave. The wave number can be determined from a nonlinear boundary conditions according to the problem statement.

5.8 Conclusions

1. We could see that Love surface waves are dispersive waves since equation (1.17) affirms that, there exists a nonlinear dependence of phase velocity v on the wave number k: meaning, when the wave number k is zero (that is, when wavelength is infinite), the phase velocity v_T^h of the transverse wave in the half-space is equal to the velocity v. Also when the wave number k is increased, velocity v decreases.

2. Both the third and the first harmonics are present in the second approximation of equations (5.33) and (5.34) as compared with equations (5.26) and (5.27) which contained only the third harmonic in z. There is a nonlinear dependence on coordinates due to the presence of amplitudes in the new harmonics which increase with increase in the time of propagation of Love wave. This causes distortion in the first harmonics.

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CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

We have considered the propagation of Love waves in layered models of the earth. The inhomogeneous isotropic model consisted of a uniform isotropic homogeneous layer overlying an inhomogeneous half-space. The perturbation and Green's function are used to obtain the dispersion relation satisfied by Love waves. We then considered the anisotropic model in which both the layer and the half-space were homogeneous anisotropic. Using the generalized Hooke's law, the governing equation was solved in the linear strain case. This model was then extended to an inhomogeneous anisotropic layer overlying an anisotropic half-space. We observed that the dispersion relation obtained in an anisotropic case gives the corresponding dispersion relation for isotropic model. The phase velocity of the Love waves was displayed graphically against wave number. We then studied the nonlinear elastic medium and presented the wave equation satisfied by the Love wave in the nonlinear case. The governing equation now involves a potential. We used the Murnaghan potential and solved the resulting nonlinear equation using the pertubation method.

6.2 Recommendations

1. The study of Love waves in nonlinear elastic model needs to be studied further. Due to the complexity of the governing equation, a numerical scheme may be developed.
2. In further studies, one may consider an inhomogeneous anisotropic layer overlying an inhomogeneous anisotropic half-space.
3. The cases of transversely isotropic and orthropic materials, which are special cases of anisotropic materials are also of interest.

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Appendix I

$$J_1 = e^{\frac{-\zeta_1 k}{\varepsilon_1}}\left(\frac{2\zeta_1 k}{\varepsilon_1}\right)^{\frac{f}{2}}, J_2 = \left(1 - \frac{(f-1)^2\varepsilon_1}{8\zeta_1 k}\right),$$

$$J_3 = \left(1 - \frac{(f+1)^2\varepsilon_1}{8\zeta_1 k} - \frac{(f-1)^2(f-2)\varepsilon_1^2}{16\zeta_1^2 k^2}\right),\ J_4 = e^{\frac{\zeta_1 k}{\varepsilon_1}}\left(\frac{-2\zeta_1 k}{\varepsilon_1}\right)^{\frac{-f}{2}},$$

$$J_5 = \left(1 + \frac{(f+1)^2\varepsilon_1}{8\zeta_1 k}\right),\ J_6 = \left(1 - \frac{(f^2+6f+1)^2\varepsilon_1}{8\zeta_1 k} - \frac{(f+1)^2(f-2)\varepsilon_1^2}{16\zeta_1^2 k^2}\right),$$

$$J_7 = \frac{-1}{2}\left(ik\alpha^2 + \frac{1}{2}\right),\ J_8 = \frac{1}{2}\left(2\zeta_1 k + \frac{i\varepsilon_1(\beta\gamma - \alpha)}{\zeta_1}\right),$$

$$J_1' = e^{\frac{-\zeta_2 k(1-\varepsilon_1 H)}{\varepsilon_1}}\left(\frac{2\zeta_2 k(1-\varepsilon_1 H)}{\varepsilon_1}\right)^{\frac{f}{2}},\ J_2' = \left(1 - \frac{(f-1)^2\varepsilon_1}{8\zeta_2 k(1-\varepsilon_1 H)}\right),$$

$$J_3' = \left(1 - \frac{(f+1)^2\varepsilon_1}{8\zeta_2 k(1-\varepsilon_1 H)} - \frac{(f-1)^2(f-2)\varepsilon_1^2}{16\zeta_2^2 k^2(1-\varepsilon_1 H)}\right),\ J_4' = e^{\frac{\zeta_2 k(1-\varepsilon_1 H)}{\varepsilon_1}}\left(\frac{-2\zeta_2 k(1-\varepsilon_1 H)}{\varepsilon_1}\right)^{\frac{-f}{2}},$$

$$J_5' = \left(1 + \frac{(f+1)^2\varepsilon_1}{8\delta_2 k(1-\varepsilon_1 H)}\right), J_6 = \left(1 - \frac{(f^2+6f+1)^2\varepsilon_1}{8\zeta_2 k(1-\varepsilon_1 H)} - \frac{(f+1)^2(f-2)\varepsilon_1^2}{16\zeta_2^2 k^2(1-\varepsilon_1 H)}\right), J_7' = \frac{-1}{2}\left(ik\alpha^2 + \frac{1}{2(1-\varepsilon_1 H)}\right),\ \text{and}\ J_8' = \frac{1}{2}\left(2\zeta_2 k + \frac{i\varepsilon_1(\beta\gamma - \alpha)}{\zeta_2(1-\varepsilon_1 H)}\right).$$

Vitae

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